Ambiguity, No Arbitrage, Coherence and Artificial Financial Markets

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Abstract

In many traditional financial models, economic agents are assumed to make decisions using expected lifetime utility under rational expectations, where rational expectations are assumed to be formed on the basis of sufficient knowledge of the data generating process. But the mere existence of econometricians modeling and estimating (risky) data generating processes already indicates the presence of ambiguity on the 'true' (possibly even non-risky) data generating process. There might be ambiguity because of sampling error or misspecification, but there might also be ambiguity due to the fact that on the basis of the available information, one may not be able to discriminate between models. Rational agents will (try to) incorporate ambiguity when making their decisions. In this paper we first investigate the implications for modeling asset prices in financial markets under the assumption of no arbitrage when there is ambiguity. We argue that coherence, as introduced by Shafer and Vovk (2001), becomes an alternative guiding principle in modeling financial markets without arbitrage opportunities. Next, we illustrate that artificial financial markets are a natural way to study coherent financial markets under ambiguity.

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1 Introduction

In many traditional financial and economic models, economic agents are assumed to make decisions using expected lifetime utility under rational expectations, where rational expectations are assumed to be formed on the basis of sufficient knowledge of the data generating process. But the mere existence of econometricians modeling and estimating (risky) data generating processes already indicates the presence of ambiguity on the 'true' (possibly even nonrisky) data generating process. There are various sources for this ambiguity. For instance, there might be ambiguity because of sampling error, arising due to the fact that one has to estimate a model, but there might also be ambiguity resulting from potential modelling error, due to a too restrictive choice of model class describing the data generating process. In this case, the model class might not fully describe all possible data generating processes, yielding possibly model misspecification. Moreover, ambiguity might arise due to the possibility that different models in a model class might describe the same data generating processes, so that, on the basis of data from a data generating process, one might not be able to discriminate between such models.

Rational agents will (try to) incorporate ambiguity when making their decisions. In this paper we investigate the implications for modelling asset prices in financial markets under the assumption of no arbitrage when there is ambiguity. Without ambiguity, i.e., when the data generating process is fully known in terms of a probability distribution describing the probabilities of the future scenarios (which we call the risky case), an arbitrage opportunity arises when a zero investment today has a nonnegative payoff tomorrow (or some other future date) with probability one, together with a strictly positive probability on a strictly positive payoff. The first fundamental theorem then yields equivalent descriptions of the assumption that there are no arbitrage opportunities that have to be satisfied by price processes in order to be arbitrage-free (see Delbaen and Schachermayer (2005)). However, when there is ambiguity on the data generating process, i.e., the probability distribution describing the probabilities of the future scenarios is ambiguous, it makes sense to redefine the concept of an arbitrage opportunity in such a context: only zero investments that yield an arbitrage opportunity according to *all* models (about which there is ambiguity) are really arbitrage opportunities. Zero investments that are arbitrage opportunities according to some models, but not to other ones, might not be real arbitrage opportunities, since the models that classify these zero investments as arbitrage opportunities might simply be incorrect. Thus, as long as there are models (about which there is ambiguity) according to which a zero investment is not an arbitrage opportunity, such an investment strategy might not be classified as an arbitrage opportunity.

Modeling the future as being described by a known probability distribution −which we call the unambiguous risky view− might alternatively be interpreted as a parsimonious description of a future that is actually behaving in a complicated, and possibly deterministic way. Perhaps such a deterministic description is so complicated, that one is unable to figure out which deterministic process −if there is one−is generating the future. In this alternative view, the used probability distribution to describe the future is just one's prior over (the outcomes of) the possible deterministic processes generating the future. If, in this view, it is assumed that some deterministic process is generating the future, there is no risk, but, if at the same time it is assumed to be unknown which deterministic process is generating the future, there is clearly ambiguity. Thus, as alternative to an unambiguous risky future we also have an ambiguous non-risky view on the future. Notice that, particularly, when the non-risky deterministic description corresponds to a dynamic system that is time-varying, it is unlikely that on the basis of data alone one will be able to distinguish (unambiguously) between the two views.

In this paper we shall consider the situation where, due to ambiguity, both the ambiguous non-risky and the unambiguous risky cases are considered as potential descriptions of the future. The 'intermediate' ambiguous risky cases are then included as well.

The assumption of no arbitrage opportunities when the ambiguous non risky cases are included as possible description of the future, corresponds to the concept of coherence, as introduced by Shafer and Vovk (2001). These authors provide a description of a financial market without probabilities. Instead of assuming that the future is predetermined up to a probability distribution, these authors view the future as an open system. In our paper we model such an open system by means of ambiguity, including at least the ambiguous non-risky case (to be included if the system is really open), but possibly also the ambiguous risky cases. To deal with an open system, Shafer and Vovk (2001) model a financial market as a game between an investor and the "market", where the "market" sets the prices, then the investor makes a portfolio investment choice, followed by new prices, set by the market, and so on. The absence of arbitrage opportunities is then translated into the requirement that the "market", as a player, and not necessarily governed by a probability distribution, sets the prices such that there are no price zero-investments generating unlimited payoffs to the investor, or stated alternatively, the "market" should always be able to guarantee a non-positive payoff for each price zero-investment. We show that the exclusion of arbitrage opportunities in an ambiguous non-risky world leads to this coherence concept.

Thus, coherence, as introduced by Shafer and Vovk (2001), might be seen as an alternative guiding principle in modelling financial markets without arbitrage opportunities, particularly when viewing the world as ambiguous and (possibly) non-risky. A natural way to model such an economy, where probabilities might not play a role, is by means of an artificial financial market, that can be investigated using microscopic simulation techniques. We describe such an artificial financial market in the tradition of, among others, Levy et al. (2000), Arthur et al. (1996), LeBaron (1999), LeBaron et al. (1999) and LeBaron (2001), but our artificial financial market explicitly allows for ambiguity with or without risk, and is constructed such that it is also coherent in the sense of Shafer and Vovk (2001).

The remainder of this paper is organized as follows. In the next section we introduce the concept of ambiguity from an econometric perspective. In particular, we describe the potential sources of ambiguity, from the viewpoint of econometric modeling. In Section 3 we illustrate ambiguity from the point of view of an individual investment problem, focussing in particular on the non-ambiguous risky case and the ambiguous non-risky case, although the ambiguous risky cases will be considered as well. We consider both "expected utility"- and mean-variance-preferences. In Section 4 we link the concept of ambiguity with arbitrage opportunities, comparing particularly the exclusion of arbitrage opportunities in the non-ambiguous risky case and the ambiguous non-risky case. Here, we also make a link with the concept of a coherent market, as introduced by Shafer and Vovk (2001). In Section 5 we introduce and describe our benchmark artificial market, which is a natural tool to investigate financial markets when there is ambiguity. Finally, Section 6 concludes.

2 Ambiguity: An Econometric Perspective

In this section we introduce ambiguity from an econometric perspective. Here, ambiguity refers to lack of knowledge concerning the data generating process. First, we introduce the concept of an econometric model, following Heckman (2000). An econometric model consists of a model class aiming at describing both data generating processes, represented by sample probability distributions, and targets, where, in our case, the targets will be probability distributions over future outcomes. Sample data is used to make a model selection, and, subsequently, to choose the appropriate target value.

In the context of econometric models we introduce the concepts of exact and overidentification at the data side and exact and underidentification at the model side. The typical econometric case is underidentification at the model side and overidentification at the data side. Both overidentification at the data side and underidentification at the model side will be sources of ambiguity: overidentification at the data side, because it might result in model misspecification, requiring a larger model class; underidentification at the model side, because it does not allow to make a data-based model choice, so that one has to deal with more than one model describing the same data, but potentially different target values. An important ingredient of an econometric model is also the selection of potential models on the basis of sample data. Particularly, when this is conducted using estimation, the presence of estimation inaccuracy will also introduce ambiguity.

Data Generating Processes
We assume that the available data is given by

$$(z_1, z_2, ..., z_T)$$

where each z_t is a k-dimensional vector containing the relevant data at time t. The data is generated by a data generating process, i.e., a (kT-dimensional) probability distribution $\mathbb{P}_T^a$, where the superscript a stands for *actual.* The set of potential probability distributions is denoted by $\mathscr{D}_T$. We assume that $(z_1, z_2, ..., z_T)$ is a realization of $(Z_1, Z_2, ..., Z_T)$, where

$$(Z_1, Z_2, ..., Z_T) \sim \mathbb{P}_T^a \in \mathscr{D}_T.$$

The set $\mathscr{D}_T$ should be large enough to include $\mathbb{P}_T^a$.[1]

Econometric Models
Next, we introduce an econometric model. This is given by the quadruple

$$\mathscr{E}_T = (\mathscr{M}, d_T, t_T, \Delta_T).$$

Here $\mathscr{M}$ denotes the model class containing various models m, for instance, depending on unknown parameter values. The models m in $\mathscr{M}$ describe potential data generating processes in $\mathscr{D}_T$. This is captured by the transformation d_T:

$$d_T : \mathscr{M} \to \mathscr{D}_T.$$

In addition, the models m in $\mathscr{M}$ also describe the target of scientific interest, as captured by the transformation t_T:

$$t_T : \mathscr{M} \to \mathscr{T}_T$$

Here, $\mathscr{T}_T$ is called the target set. In our case, we shall consider as target values potential probability distributions of the payoffs of J assets at some future time

[1] Notice that $\mathscr{D}_T$ might contain degenerate probability distributions.

$T+1$, say, conditional upon the (model specific) available information at time T. Finally, Δ_T is a data selection procedure, given by

$$\Delta_T = \Delta_T(z_1, ..., z_T) \subset \mathscr{D}_T.$$

Usually, Δ_T is the outcome of an estimation procedure, for instance, a point estimate or a confidence interval around some point estimate in case of estimation inaccuracy. But alternative data selection procedures, like calibration based ones, are, of course, also possible. Figure 1 illustrates the set-up.

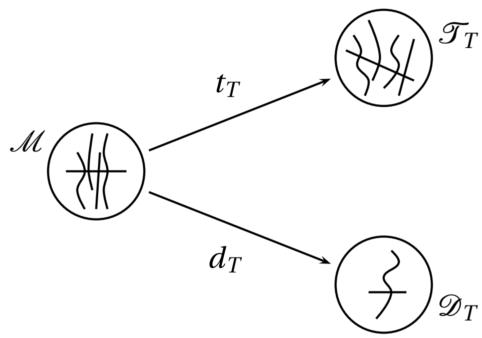

Figure 1: Ambiguity

Identification
Given an econometric model class, we have *identification at the data side* when

$$d_T(\mathscr{M}) = \mathscr{D}_T,$$

and we have *overidentification at the data side* when

$$d_T(\mathscr{M}) \subsetneq \mathscr{D}_T.$$

Thus, model class $\mathscr{M}$ is identified at the data side in case it is large enough to describe all possible data. Otherwise, when the model class $\mathscr{M}$ is overidentified at the data side, the model class does not describe all possible data, so that there is the possibility of *misspecification*, i.e., we might have that

$$\mathbb{P}_T^a \in \mathscr{D}_T \backslash d_T(\mathscr{M}).$$

We have *identification at the model side*, when

$$d_T \text{ is one-to-one (injective),}$$

and we have *underidentification at the model side* when

$$d_T \text{ is not one-to-one.}$$

Thus, model class $\mathscr{M}$ is identified at the model side when for each data point in $\mathbb{P}_T \in d_T(\mathscr{M})$, there is exactly one model $m \in \mathscr{M}$, such that $d_T(m) = \mathbb{P}_T$. Otherwise, when the model class $\mathscr{M}$ is underidentified at the model side, there are data points $\mathbb{P}_T \in d_T(\mathscr{M})$ which cannot be described by a single model $m \in \mathscr{M}$. Instead, we have models $m_1, m_2 \in \mathscr{M}$, with $m_1 \neq m_2$, such that $d_T(m_1) = d_T(m_2) = \mathbb{P}_T$. In this case the data does not allow us to discriminate between models m_1 and m_2.

Ambiguity
Given Δ_T, we select as models

$$d_T^{-1}(\Delta_T) = \{m \in \mathscr{M} : d_T(m) \in \mathscr{D}_T\} \subset \mathscr{M},$$

resulting in the selected target values

$$\left(t_T \circ d_T^{-1}\right)(\Delta_T) = \left\{t_T(m) \in \mathscr{T}_T : m \in d_T^{-1}(\Delta_T)\right\}.$$

There is *no ambiguity* when this set of selected target values consists of a singleton. Otherwise, when the set of target values contains more than one element, there is *ambiguity*.

Sources of Ambiguity
There are many sources of ambiguity. The standard econometric approach consists of choosing a model class $\mathscr{M}$, that is usually exactly identified at the model side, but overidentified at the data side. For instance, one models the asset returns as independent over time and identically distributed according to a normal distribution, parameterized by the mean vector and covariance matrix. Then, given a possible normal distribution function in $\mathscr{D}_T$, one is able to retrieve the model in the model class, i.e., there is identification at the model side. But, likely, asset returns are not normal, or not independent and identically distributed, so that at the data side we might possibly have misspecification, i.e., overidentification. Based on the model class, one selects some estimation procedure, like Maximum Likelihood, that generally does not only result in a single point estimate in $\mathscr{D}_T$ but also in a confidence set around this point estimate. In this case it makes sense to choose for Δ_T the confidence set in $\mathscr{D}_T$. Due to this estimation inaccuracy we cannot be sure about the model in the model class $\mathscr{M}$, and we are dealing with ambiguity (unless all models in $d_T^{-1}(\Delta_T)$ would have the same target value, which is not very likely).

Misspecification, due to overidentification at the data side, is an additional source of ambiguity. Generally, to be sure that $\mathscr{D}_T$ really includes $\mathbb{P}_T^a$ we have to choose $\mathscr{D}_T$ so large, that the typical econometric model does not provide a description for the whole set $\mathscr{D}_T$. If we are not sure that we exclude potential misspecification, then we might want to enlarge the model class $\mathscr{M}$, resulting in a more extensive description of the data generating processes in $\mathscr{D}_T$. But, generally, this also leads to a larger data selection set Δ_T, so that, as a consequence, the set of target values might become larger, and, thus, increasing the ambiguity. Misspecification tests might be of some help in deciding whether the chosen model class $\mathscr{M}$ is large enough to yield a sufficiently broad description of the set $\mathscr{D}_T$ to avoid misspecification.

Underidentification at the model side is an often overlooked source of ambiguity. Although usually one *chooses* a model class that is exactly identified at the model side, since otherwise estimation of a model in such a model class is not feasible, this does not exclude the possibility that there are other model classes that describe the same data generating processes, but that result in different sets of target values than the original model class, one starts with. When such alternative model classes indeed do exist, econometrics cannot provide −by definition− data based methods to make a choice between these alternative model classes. But, of course, such alternative model classes would introduce ambiguity.

3 Illustration: Investment Decisions

Consider an investor who wants to choose in an optimal way his or her portfolio, consisting of an investment in $J+1$ assets, numbered from 0 to J, where asset 0 always has price 1 and dividend payoff 0 ("money" or the risk free asset with interest payment equal to zero). Let $X_t = S_t + D_t$ stand for the vector of total payoff of the J assets at time t, with S_t the vector of asset prices at time t, and with D_t the corresponding vector of dividend payoffs at time t. For simplicity, we consider a one-period ahead portfolio choice problem at time $t = T$. Let $\mathscr{X}_{T+1} \subset \mathbb{R}^{J+1}$ denote the set of all possible values of the assets' payoff vector X_{T+1} at time $T+1$. We shall assume that this set $\mathscr{X}_{T+1}$ is finite. A portfolio at time T is given by a vector $h_T \in \mathbb{R}^{J+1}$, with time T portfolio price given by $h_T \cdot S_T$, and its total payoff at time $T+1$ equal to

$$h_T \cdot X_{T+1} = h_T \cdot (S_{T+1} + D_{T+1})$$

(where a $\cdot$ means taking an inner product). We assume that the investor will make the investment decision h_T given some econometric model and given some preference ordering. We shall first illustrate three cases in the context of expected utility: the unambiguous case, the ambiguous case due to estimation inaccuracy, and the ambiguous case due to underidentification, but without estimation inaccuracy. Then we also briefly illustrate in the context of mean-variance preferences.

1. *The unambiguous risky case*
The unambiguous case corresponds to the situation in which the investor has an econometric model

$$\mathscr{E}_T = (\mathscr{M}, d_T, t_T, \Delta_T).$$

such that for some $m_0 \in \mathscr{M}$

$$\{m_0\} = d_T^{-1}(\Delta_T),$$

with target value

$$\mathbb{P}_{m_0} \equiv t_T(m_0),$$

where the target value $\mathbb{P}_{m_0} \equiv t_T(m_0)$ is the conditional probability distribution of X_{T+1}, given the available model m_0 specific information up to and including time T, with support contained in $\mathscr{X}_{T+1}$. Assume that the investor has utility index $U_{T+1} : \mathscr{X}_{T+1} \to \mathbb{R}$, and is an expected utility optimizer. Then the optimal investment problem can be formulated as

$$\begin{aligned} \max_{h_T} \mathbb{E}_{m_0}(U_{T+1}(h_T \cdot X_{T+1})) &= \max_{h_T} \int_{\mathscr{X}_{T+1}} U_{T+1}(h_T \cdot x)\, d\mathbb{P}_{m_0}(x) \\ s.t.\ h_T \cdot S_T &= W_T, \end{aligned}$$

with $\mathbb{E}_{m_0}$ denoting taking expectations with respect to $\mathbb{P}_{m_0}$, and with W_T the wealth available for investment. In the sequel, we shall refer to this case as the unambiguous risky case.

2. *Ambiguity due to estimation inaccuracy*
Ambiguity arises as soon as

$$\mathscr{M}_0 = d_T^{-1}(\Delta_T)$$

is a subset of the model class $\mathscr{M}$, containing more than one element, such that there exist $m_1,\ m_2 \in \mathscr{M}_0$ satisfying

$$\mathbb{P}_{m_1} \equiv t_T(m_1) \neq t_T(m_2) \equiv \mathbb{P}_{m_2}.$$

As discussed in the previous section, ambiguity might arise due to estimation inaccuracy in which case Δ_T is not a point estimate but, for instance, a confidence interval around some point estimate. To deal with such a case, we can follow Klibanoff et al. (2005). Let π denote a "prior" distribution over $\mathscr{M}$, with support contained in $\mathscr{M}_0$, and suppose that the investor is not only characterized by the utility index U_{T+1}, but also by an ambiguity index

$$\phi_{T+1} : \mathbb{R} \to \mathbb{R}.$$

Then we can formulate the optimal investment problem as

$$\begin{aligned} & \max_{h_T} \mathbb{E}_\pi \left(\phi_{T+1} \left(\mathbb{E}_m \left(U_{T+1} \left(h_T \cdot X_{T+1} \right) \right) \right) \right) \\ = \; & \max_{h_T} \int_{\mathscr{M}_0} \phi_{T+1} \left(\int_{\mathscr{X}_{T+1}} U_{T+1} \left(h_T \cdot x \right) d\mathbb{P}_m (x) \right) d\pi (m) \\ & s.t. \; h_T \cdot S_T = W_T, \end{aligned}$$

where $\mathbb{E}_\pi$ denotes taking expectations with respect to π . We shall summarize this optimal investment problem by

$$\left(U_{T+1}, \phi_{T+1}, \pi \right),$$

and refer to it as the estimation-ambiguous risky case.

The unambiguous case, as discussed under 1, corresponding to a specific model $m_0 \in \mathscr{M}$, can then be summarized as

$$\left(U_{T+1}, I, \delta_{m_0} \right),$$

where the prior $\pi = \delta_{m_0}$ is the one-point mass distribution that assigns unit mass to model m_0, and where the ambiguity index $\phi_{T+1} = I$ is the identity transformation from $\mathbb{R}$ to $\mathbb{R}$.

3. *Ambiguity due to underidentification without estimation inaccuracy*
Ambiguity need not result from estimation inaccuracy. An important reason why econometric models –describing probability distributions in $\mathscr{D}_T$– are employed, seems to be lack of knowledge. Instead of considering the risky unambiguous case, as in 1., we might consider as alternative the possibility that a description of the economy is obtained by assuming that the economy is driven by a (possibly) complicated, but deterministic process in terms of some vector of state variables w_t, governed by dynamics described by $w_t = \Phi_t^m (w_{t-1})$, while we observe $z_t = f_t^m (w_t)$, $t = 1, ..., T$, and we are interested in $X_{T+1} = g_{T+1}^m (w_{T+1})$, where the functions Φ_t^m, f_t^m, and g_{T+1}^m are model m specific, for some given model m. The data is then postulated to be generated by the one-point mass distribution $\delta^m_{(z_1,...,z_T)}$, as induced by model m, in case $(Z_1, ..., Z_T) = (z_1, ..., z_T)$. However, we might not know which deterministic process is generating the economy. So, we might want to allow for all such possibilities. If we allow for such a deterministic description of the economy, and we want to avoid

potential misspecification, we need to allow for the possibility that $\mathscr{D}_T$ includes the set $\mathscr{P}_T$ of one-point mass distributions, and we have to assume that the model class $\mathscr{M}$ is rich enough to include as subset a subclass $\mathscr{M}_{\mathscr{P}}$ describing all one-point mass distributions of

$$(Z_1, Z_2, ..., Z_T, X_{T+1}).$$

Let

$$d_T : \mathscr{M}_{\mathscr{P}} \to \mathscr{P}_T \subset \mathscr{D}_T$$

assign to each $m \in \mathscr{M}_{\mathscr{P}}$ the one-point mass distribution $\delta^m_{(z_1, z_2, ..., z_T)}$ of

$$(Z_1, Z_2, ..., Z_T)$$

according to model m. Assume $\Delta_T = \left\{\delta_{(z_1, z_2, ..., z_T)}\right\}$, i.e., we select as data the one-point mass distribution corresponding to the observed data $(z_1, z_2, ..., z_T)$. Obviously, there is *no* estimation inaccuracy in this data selection. Next, let

$$\mathscr{M}_0 = d_T^{-1}\left(\left\{\delta_{(z_1, z_2, ..., z_T)}\right\}\right),$$

then $\mathscr{M}_0 \subset \mathscr{M}$ includes those models in the subclass $\mathscr{M}_{\mathscr{P}}$ of one-point mass distributions of $(Z_1, Z_2, ..., Z_T, X_{T+1})$ that satisfy $(Z_1, Z_2, ..., Z_T) = (z_1, z_2, ..., z_T)$. Clearly, if $\mathscr{M}$ is so rich that it includes the subclass $\mathscr{M}_{\mathscr{P}}$, then we are dealing with the case of underidentification. When determining the target values of $\mathscr{M}_0$ we find

$$t_T(\mathscr{M}_0) \supset \{\delta_x : x \in \mathscr{X}_{T+1}\},$$

i.e., target values include at least the set of all one-point mass distributions over $\mathscr{X}_{T+1}$. There is clearly ambiguity, but due to underidentification, and not due to estimation inaccuracy.

To model the optimal investment problems, we can still use Klibanoff et al. (2005). Suppose that $\mathscr{M}_0 \subset \mathscr{M}_p$ and $t_T(\mathscr{M}_0) = \{\delta_x : x \in \mathscr{X}_{T+1}\}$, so that we can identify to each $x \in \mathscr{X}_{T+1}$, corresponding to the one-point mass distribution δ_x, a model $m \in \mathscr{M}_0 \subset \mathscr{M}_p$. Using this identification of each $x \in \mathscr{X}_{T+1}$ with some model $m \in \mathscr{M}_0 \subset \mathscr{M}_p$, we can consider the prior distribution π also as being defined over $\mathscr{X}_{T+1}$. To indicate this, we write $\pi_{\mathscr{X}_{T+1}}$. In addition, given a model $m \in \mathscr{M}_0 \subset \mathscr{M}_p$, there is no risk, so we could take a linear utility index. Thus, in case of Klibanoff et al. (2005), we have as objective function

$$\mathbb{E}_{\pi_{\mathscr{X}_{T+1}}}\left(\phi_{T+1}\left(\mathbb{E}_{\delta_x}(h_T \cdot X_{T+1})\right)\right) = \int_{\mathscr{X}_{T+1}} \phi_{T+1}(h_T \cdot x)\, d\pi_{\mathscr{X}_{T+1}}(x).$$

The optimal investment problem becomes

$$\max_{h_T} \int_{\mathscr{X}_{T+1}} \phi_{T+1}(h_T \cdot x)\, d\pi_{\mathscr{X}_{T+1}}$$

$$s.t.\ h_T \cdot S_T = W_T.$$

We can summarize this optimization problem as

$$\left(I, \phi_{T+1}, \pi_{\mathscr{X}_{T+1}}\right).$$

where the prior distribution $\pi_{\mathscr{X}_{T+1}}$ only assigns (positive or zero) probability mass to models $m \in \mathscr{M}_0 \subset \mathscr{M}_p$ that correspond to one-point mass distributions δ_x, $x \in \mathscr{X}_{T+1}$. In the sequel we shall refer to this case as the ambiguous non-risky case.

The ambiguous risky case can be obtained by integrating cases 2. and 3. We are then dealing with random dynamic systems, as studied, for instance, in Arnold (2003). In the sequel, we shall mainly focus on the unambiguous risky case and the unambiguous non-risky case.

4. *Comparison of nonambiguous risk and ambiguous non-risk*

Comparing cases 1, the unambiguous risky case, and 3, the ambiguous non-risky case, we see that these become indistinguishable (from the point of view of the investment decision) as soon as we make the identification

$$\phi_{T+1} \leftrightarrow U_{T+1},\ \pi_{\mathscr{X}_{T+1}} \leftrightarrow \mathbb{P}_{m_0},$$

i.e.,

$$\left(U_{T+1}, I, \delta_{m_0}\right) \Longleftrightarrow \left(I, U_{T+1}, \mathbb{P}_{m_0}\right).$$

This yields the following possible interpretation, particularly, when in case of 1 one uses for model m_0 an estimated model, say $\widehat{m}$, and one does not employ 2, explicitly taking into account estimation inaccuracy, but one ignores estimation inaccuracy. Then, instead of dealing with the optimization problem summarized by $(U_{T+1}, I, \delta_{\widehat{m}})$, one could equivalently claim to be dealing with $(I, U_{T+1}, \mathbb{P}_{\widehat{m}})$, where the empirically based probability distribution $\mathbb{P}_{\widehat{m}}$ is not used to describe one's expected utility (which is indeed hard to defend if there is estimation ambiguity), but, instead, $\mathbb{P}_{\widehat{m}}$ –based on historical data– is used to describe one's prior over ambiguous models $m \in \mathscr{M}_0 \subset \mathscr{M}_p$ corresponding to one point mass distributions over $\mathscr{X}_{T+1}$. Which case makes most sense is obviously quite ambiguous.

4. *Mean-Variance Preferences*

The two cases 1. and 3. can also be compared in terms of mean-variance preferences. In case 1., with model m_0, the investor's optimal investment problem can be formulated as

$$\max_{h_T} \mathbb{E}_{m_0}\left(h_T \cdot X_{T+1}\right) - \frac{\gamma}{2} Var_{m_0}\left(h_T \cdot X_{T+1}\right)$$

$$s.t.\ h_T \cdot S_T = W_T,$$

where $\mathbb{E}_{m_0}$ and Var_{m_0} denote taking the expectation and variance using $\mathbb{P}_{m_0}$, respectively, and where γ represents the risk aversion parameter of the investor. Suppose that the investor employs some estimated model: $m_0 = \widehat{m}$. Then an alternative interpretation is to interpret the investor's preferences as a variational preference under ambiguity, following Maccheroni et al. (2004). Translated to the current context, these authors assume the presence of an ambiguity index function

$$c\left(\cdot \parallel \mathbb{P}_{\widehat{m}}\right) : \mathscr{P}\left(\mathscr{X}_{T+1}\right) \to \mathbb{R},$$

where $\mathscr{P}\left(\mathscr{X}_{T+1}\right)$ stands for the set of all probability distributions over $\mathscr{X}_{T+1}$, defined by

$$c\left(\mathbb{P}_m \parallel \mathbb{P}_{\widehat{m}}\right) = \int \left(\frac{d\mathbb{P}_m}{d\mathbb{P}_{\widehat{m}}}\right)^2 d\mathbb{P}_{\widehat{m}} - 1,$$

if $\mathbb{P}_m$ is absolutely continuously with respect to $\mathbb{P}_{\widehat{m}}$, and

$$c\left(\mathbb{P}_m \parallel \mathbb{P}_{\widehat{m}}\right) = +\infty,$$

otherwise. Let

$$V_T\left(h_T\right) = \min_{\mathbb{P}_m \in \mathscr{P}^{\sigma}\left(\mathbb{P}_{\widehat{m}}\right)} \left(\mathbb{E}_m\left(h_T \cdot X_{T+1}\right) - \frac{1}{2\gamma} c\left(\mathbb{P}_m \parallel \mathbb{P}_{\widehat{m}}\right)\right),$$

where $\mathscr{P}^{\sigma}\left(\mathbb{P}_{\widehat{m}}\right)$ denotes the set of probability distributions over $\mathscr{X}_{T+1}$ that are absolutely continuous with respect to $\mathbb{P}_{\widehat{m}}$. Preferences that can be represented this way, are called variational preferences (under ambiguity) by Maccheroni et al. (2004). These authors present preference axioms leading to such a representation (and more general ones, than the one presented here). Moreover, under appropriate regularity conditions, these authors show the equivalence

$$\begin{array}{rcl} \mathbb{E}_{\widehat{m}}\left(h_T \cdot X_{T+1}\right) - \frac{\gamma}{2} Var_{\widehat{m}}\left(h_T \cdot X_{T+1}\right) & \geq & \mathbb{E}_{\widehat{m}}\left(g_T \cdot X_{T+1}\right) - \frac{\gamma}{2} Var_{\widehat{m}}\left(g_T \cdot X_{T+1}\right) \\ & \Longleftrightarrow & \\ V_T\left(h_T\right) & \geq & V_T\left(g_T\right). \end{array}$$

Thus, mean variance preferences can also be viewed as variational preferences, explicitly dealing with ambiguity.

4 Ambiguity and No Arbitrage

In this section we investigate various types of arbitrage opportunities. For simplicity, we focus on the single-period context, where the present is time T, and the next period is time $T+1$, and where we have available an econometric model

$$\mathscr{E}_T = (\mathscr{M}, d_T, t_T, \Delta_T).$$

Similar to the previous section, let $X_t = S_t + D_t$ stand for the vector of total payoff of the $J+1$ assets at time t, with $S_t \in \mathbb{R}^{J+1}$ the vector of asset prices at time t, and with $D_t \in \mathbb{R}^{J+1}$ the corresponding vector of dividend payoffs at time t. Again, let $\mathscr{X}_{T+1} \subset \mathbb{R}^{J+1}$ denote the set of all possible values of the assets' payoff vector X_{T+1} at time $T+1$, which we also in this section assume to be finite. A portfolio at time T is given by a vector $h_T \in \mathbb{R}^{J+1}$, so that short-selling is allowed, with time T portfolio price given by $h_T \cdot S_T$, and its total payoff at time $T+1$ equal to

$$h_T \cdot X_{T+1} = h_T \cdot (S_{T+1} + D_{T+1}).$$

For a specific $m \in \mathscr{M}$, let the target $\mathbb{P}_m = t_T(m)$ be the conditional probability distribution of X_{T+1}, given the available −model m specific− information up to and including time T, with support contained in $\mathscr{X}_{T+1}$.

We shall distinguish between the following types of arbitrage opportunities.

Definition 1 *Portfolio h_T (with price $h_T \cdot S_T = 0$) represents a model m specific arbitrage opportunity at time T in case*

$$\mathbb{P}_m(h_T \cdot X_{T+1} \geq 0) = 1 \text{ and } \mathbb{P}_m(h_T \cdot X_{T+1} > 0) > 0.$$

Definition 2 *Portfolio h_T (with price $h_T \cdot S_T = 0$) represents a model subclass $\mathscr{N} \subset \mathscr{M}$ specific arbitrage opportunity at time T, in case portfolio h_T represents a model m specific arbitrage opportunity at time T for all $m \in \mathscr{N}$.*

Definition 3 *Portfolio h_T (with price $h_T \cdot S_T = 0$) represents a data Δ_T specific arbitrage opportunity at time T, in case portfolio h_T represents a model subclass $d_T^{-1}(\Delta_T) \subset \mathscr{M}$ specific arbitrage opportunity at time T.*

Definition 1 is the usual definition of no arbitrage opportunities in an unambiguous risky setting (see Delbaen and Schachermayer (2005)). However, as soon as there is ambiguity in the sense that all we now is that a model in $\mathscr{N}$ results in an appropriate description of the next period, it makes sense to consider definition 2. In case one makes use of an econometric model, it makes sense to exclude arbitrage opportunities according to definition 3.

Assuming that there are no arbitrage opportunities, means that particular prices S_T have to be excluded. To illustrate that definition 1 and 2 generally do not result in the same restrictions on possible prices S_T when arbitrage opportunities are excluded, consider the following very simple, but illustrative example.

Example 4 *Let $J = 2$ with*

$$\mathscr{X}_{T+1} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\}$$

and with

$$S_T = \begin{pmatrix} 1 \\ S_{2,T} \end{pmatrix}.$$

Consider as models

$$\mathbb{P}_{m_p} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = p, \ \mathbb{P}_{m_p} \left\{ \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\} = 1 - p.$$

If $\mathscr{N} = \{m_p : \ p \in (0,1)\}$ then $S_{2,T} \notin \left(\frac{1}{2}, 2\right)$ result in model class $\mathscr{N}$-specific arbitrage opportunities. However, if $\mathscr{N}$ also includes m_1 and m_0, then $S_{2,T} = 2$ or $S_{2,T} = \frac{1}{2}$ do no longer result in model class $\mathscr{N}$-specific arbitrage opportunities.

Notice that model m_p, with $p \in (0,1)$ corresponds to an unambiguous risky modelling of the future (see figure 2), while $\{m_0, m_1\}$, with m_1 having prior p and

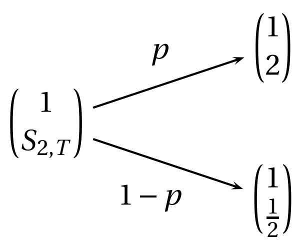

Figure 2: Unambiguous, risky interpretation

m_0 having prior $1 - p$ corresponds to an ambiguous, non-risky view of the same future (see figure 3). Modeling the future by means of risk, i.e., assuming that

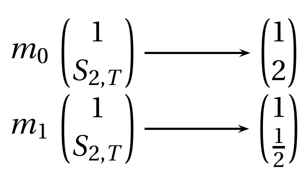

Figure 3: Ambiguous, non-risky interpretation

the future is described by some m_p, $p \in (0,1)$, means that, in case one excludes arbitrage opportunities, one has to exclude as possible prices $S_{2,T} = 2$ or $S_{2,T} = \frac{1}{2}$. Alternatively, if one models the future under ambiguity, i.e., one considers as possible models m_0 and m_1, then, in case one excludes arbitrage opportunities, *price $S_{2,T} = 2$ is not excluded, but, instead, price $S_{2,T} = 2$ will exclude model m_0 in case one would investigate choosing a trading strategy aiming at exploiting an arbitrage opportunity according to model m_0. Similarly, price $S_{2,T} = \frac{1}{2}$ is not excluded, but it might exclude model m_1.*

In terms of priors $\pi_{\mathscr{X}_{T+1}}$, consider an investor who is a price-taker. Then, as long as $S_{2,T} \in \left(\frac{1}{2}, 2\right)$, any prior with support $\mathscr{X}_{T+1}$ makes sense. But when $S_{2,T} = 2$, the assumption that there are no arbitrage opportunities, implies that only the prior $\pi_{\mathscr{X}_{T+1}} = \delta_{\{2\}}$ makes sense as soon as the investor would investigate a trading strategy aiming at exploiting an arbitrage opportunity according to model m_0. Similarly, when $S_{2,T} = \frac{1}{2}$, only the prior $\pi_{\mathscr{X}_{T+1}} = \delta_{\{\frac{1}{2}\}}$ makes sense under the assumption that there are no arbitrage opportunities and the investor would investigate a trading strategy aiming at exploiting an arbitrage opportunity according to model m_1.

As discussed in the previous section, the unambiguous risky case with probability distribution estimated, is indistinguishable from the ambiguous non-risky case when we identify the utility index of the risky model with the ambiguity index of the ambiguous model, and use in the ambiguous case as prior the estimated probability distribution. As illustrated in the previous example these different interpretations might correspond to different classifications of arbitrage opportunities. The next theorem characterizes arbitrage opportunities in the ambiguous non-risky case.

Theorem 5 *Let model class $\mathscr{M}$ include as subset the subclass $\mathscr{M}_{\mathscr{P}}$ describing all point mass distributions of*

$$(Z_1, Z_2, ..., Z_T, X_{T+1}).$$

Let

$$d_T : \mathscr{M}_{\mathscr{P}} \to \mathscr{P}_T \subset \mathscr{D}_T$$

assign to each $m \in \mathscr{M}_{\mathscr{P}}$ the point mass distribution $\delta^m_{(z_1, z_2, ..., z_T)}$ of

$$(Z_1, Z_2, ..., Z_T)$$

according to model m. Assume $\Delta_T = \left\{\delta_{(z_1, z_2, ..., z_T)}\right\}$, and let

$$\mathscr{M}_0 = d_T^{-1}\left(\left\{\delta_{(z_1, z_2, ..., z_T)}\right\}\right),$$

and

$$t_T(\mathscr{M}_0) = \{\delta_x : x \in \mathscr{X}_{T+1}\},$$

i.e., the set of all point mass distributions over $\mathscr{X}_{T+1}$*. Then portfolio* h_T *(with price* $h_T \cdot S_T = 0$*) represents a data* Δ_T *specific arbitrage opportunity at time* T *if and only if*

$$h_T \cdot x > 0$$

for all $x \in \mathscr{X}_{T+1}$*.*

Proof. If $h_T \cdot x > 0$ for all $x \in \mathscr{X}_{T+1}$ then each model m corresponding to some δ_x, with $x \in \mathscr{X}$, generates an arbitrage opportunity. This shows that portfolio h_T (with price $h_T \cdot S_T = 0$) is a data Δ_T specific arbitrage opportunity at time T. Conversely, if portfolio h_T (with price $h_T \cdot S_T = 0$) represents a data Δ_T specific arbitrage opportunity at time $T+1$, then each model m corresponding to some δ_x, with $x \in \mathscr{X}_{T+1}$, generates an arbitrage opportunity. But this is only possible if $h_T \cdot x > 0$ for all $x \in \mathscr{X}_{T+1}$. ■

Thus, in order to avoid arbitrage opportunities in the ambiguous non-risky case, we need to avoid prices S_T in period T and payoffs X_{T+1} in period $T+1$ that allow for trading strategies with zero price in period T but always positive payoff in period $T+1$. In other words, for any trading strategy that is costless in period T, there should be at least one nonpositive payoff in period $T+1$. This latter interpretation corresponds to a financial market game, as introduced by Shafer and Vovk (2001). These authors investigate an investor playing against the "market", where, in period T, the market first sets S_T, then the investor chooses a portfolio strategy h_T, while in the following period $T+1$, the market selects S_{T+1} and D_{T+1}, and, thus, $X_{T+1} = S_{T+1} + D_{T+1}$. Then the investor chooses portfolio h_{T+1}, and so on. The assumption that there are no arbitrage opportunities in the ambiguous non-risky case corresponds to the requirement of market coherence as imposed by Shafer and Vovk (2001): the market should be able to set the prices S_T in period T and payoffs X_{T+1} in period $T+1$, such that there are no zero-investment strategies that would generate unlimited gains to the investor, i.e., the market should be able to guarantee a non-positive payoff for each zero-investment strategy.

As corollary of the theorem we present an easy sufficient condition by which arbitrage opportunities under ambiguity but without risk will be excluded.

Corollary 6 *Let* $S_T \in \mathscr{X}_{T+1}$*, then there are no data* $\Delta_T = \left\{\delta_{(z_1,z_2,\ldots,z_T)}\right\}$ *specific arbitrage opportunities at time* T*.*

Proof. If $h_T \cdot S_T = 0$, then $h_T \cdot x = 0$ for $x = S_T \in \mathscr{X}_{T+1}$, and $h_T \cdot x > 0$ for all $x \in \mathscr{X}_{T+1}$ is excluded. ■

In the unambiguous risky case an arbitrage opportunity arises as soon as a zero investment at time T has a positive payoff in only one future scenario in period $T+1$, with zero payoff otherwise. In the ambiguous non-risky case arbitrage opportunities will already be excluded when a zero investment at time T has a zero payoff in only one future scenario in period $T+1$. What this corollary tells us, is that finding such a single scenario might not be hard. Indeed, quite often today's prices make sense as tomorrow's potential payoffs. Particularly, when stock prices and payoffs are modeled, this corollary indicates that, generally speaking, the exclusion of arbitrage opportunities in the ambiguous non-risky case is to be expected.

More generally, we have the following corollary.

Corollary 7 *Suppose there exists some non-zero transformation $m : \mathscr{X}_{T+1} \to \mathbb{R}$, such that for all $x \in \mathscr{X}_{T+1}$ we have $m(x) \geq 0$, and such that for all $h_T \in \mathbb{R}$ we have $h_T \cdot S_T = \sum_{x \in \mathscr{X}_{T+1}} m(x)\,(h_T \cdot x)$. Then there are no data $\Delta_T = \left\{\delta_{(z_1, z_2, \ldots, z_T)}\right\}$ specific arbitrage opportunities at time T.*

Notice that the existence of a transformation $m : \mathscr{X}_{T+1} \to \mathbb{R}$ actually follows by the Law of One Price, which is implied by the theorem.

Proof. Given a non-zero m, satisfying $m(x) \geq 0$ for all $x \in \mathscr{X}_{T+1}$, we also have for some h_T such that $h_T \cdot x > 0$, for all $x \in \mathscr{X}_{T+1}$,

$$h_T \cdot S_T = \sum_{x \in \mathscr{X}_{T+1}} m(x)\,(h_T \cdot x) > 0.$$

Thus, data $\Delta_T = \left\{\delta_{(z_1, z_2, \ldots, z_T)}\right\}$ specific arbitrage opportunities at time T are excluded. ■

Excluding arbitrage opportunities on $\mathscr{X}_{T+1}$ in the unambiguous risky case (assuming that each $x \in \mathscr{X}_{T+1}$ has a strictly positive probability) requires the existence of some transformation $m : \mathscr{X}_{T+1} \to \mathbb{R}$ that is strictly positive everywhere. In the ambiguous non-risky case, a non-zero non-negative transformation $m : \mathscr{X}_{T+1} \to \mathbb{R}$ already suffices to exclude $\Delta_T = \left\{\delta_{(z_1, z_2, \ldots, z_T)}\right\}$ specific arbitrage opportunities. The interpretation is that there exists a subset of the set of potential outcomes $\mathscr{X}_{T+1}$, determined by the requirement $m(x) > 0$ such that on this subset, when considered as an unambiguous non-risky modeling of the future, arbitrage opportunities from that point of view are excluded.

In the next section we shall use the corollaries to construct a coherent artificial stock market model. Such coherent models can be studied using the theory developed by Shafer and Vovk (2001).

5 A coherent artificial financial stock market model

In this section we illustrate how an ambiguous, possibly non-risky economy can be modeled in a natural way as an artificial financial market. Adriaens et al.

(2006b) contains a more extensive investigation. Since we will not be able to solve the dynamics of this artificial financial market analytically, we will have to use microscopic simulation to simulate the evolution of the market over time. From that point of view, the set-up chosen here can be seen as an alternative to the work by, among others, Levy et al. (2000), Arthur et al. (1996), LeBaron (1999), LeBaron et al. (1999) and LeBaron (2001). In these microscopic simulation models, economic agents are usually subdivided into various types, such as technical analysts or believers of the Efficient Market Hypothesis, and so on. In our economy, the agents are all equally "rational" in the sense of utility maximization or mean-variance optimization (either unambiguous risky or ambiguous non-risky), but different in the way they quantify the future, due to different econometric models employed. In this section we present a *benchmark* version of such an economy.

We follow the economy during periods $t \in \{1, 2, \ldots, T\}$. In this economy there is one numeraire asset, "money" (or the "riskfree asset"), whose price ($S_{0,t}$) in all periods equals 1, and whose dividend payoffs ($D_{0,t}$) always equals 0. In addition, there are J assets, where asset j is characterized by its dividend payoff D_{jt} and price S_{jt} in period t. There are J "firms" in the economy, which decide on the dividend payoffs, where "firm" j decides on D_{jt} in period t. The economy is inhabited by I investors, where each period t investor i chooses the portfolio holdings

$$h_t^i = \left(h_{0,t}^i, h_{1,t}^i, ..., h_{J,t}^i\right)$$

where $h_{j,t}^i$ denotes the portfolio holding of agent i in asset j as decided upon in period t. When entering period t the portfolio holdings of investor i are given by

$$h_{t-1}^i = \left(h_{0,t-1}^i, h_{1,t-1}^i, \ldots, h_{J,t-1}^i\right),$$

i.e., the portfolio holdings as chosen in the previous period. These investors trade at $J+1$ markets, one for each of the $J+1$ assets. At each market there are a regulator and a market maker. The market maker's task is to set the price of asset j at time t such that the total supply equals the total demand, i.e., without intervention of the regulator, the price S_{jt} should be set such that

$$\sum_{i=1}^{I} h_{j,t}^i = \sum_{i=1}^{I} h_{j,t-1}^i.$$

However, the regulators, responsible for an "orderly behavior" of the financial markets might intervene in case the resulting price, as is to be set by the market makers, is considered to be too "disorderly". The regulators might intervene by active trading (buying or selling assets), in which case the market makers also incorporate this additional demand or supply in the price setting, or the regulators might intervene by imposing a price change limit, in which case the market makers take care of an appropriate rationing.

The timing is as follows. At time t first each of the firms $j = 1,\dots,J$ decides on its dividend payoff D_{jt}, then the investors $i = 1,\dots,I$ reveal their demands for each of the assets $j = 1,\dots,J$, and the market maker of market j determines the price, at which demand equals supply. Then the regulator of market j intervenes or not, and, subsequently, the price S_{jt} is set according to the market rules and the new portfolio holdings h^i_{jt} of investor i in terms of asset j are determined.

The "firms" are responsible for issuing dividends. In financial models, the dividend process is usually assumed to be exogenously given. We follow this tradition. In the benchmark version of the economy, we simply assume that each of the "firms" uses some (possibly deterministic) rule for determining the dividend payment in each period. This rule might be varying over time and is unknown to the other economic agents.

Investor i is assumed to employ in period t an econometric model

$$\mathscr{E}^{i,t}_T = \left(\mathscr{M}^{i,t}, d^{i,t}_T, t^{i,t}_T, \Delta^{i,t}_T\right),$$

where $T = T^{i,t}$ is the individual and time specific memory length. In the benchmark version of the economy the investors only consider a one-step ahead investment choice. Assuming "expected utility", investor i is characterized in period t by

$$\left(U^{i,t}_{t+1}, \phi^{i,t}_{t+1}, \pi^{i,t}\right).$$

with $\pi^{i,t}$ having support contained in

$$\mathscr{M}^{i,t}_0 = \left(d^{i,t}_T\right)^{-1}\left(\Delta^{i,t}_T\right).$$

Thus, investor i is assumed to solve in period t

$$\max_{h^i_t} \int_{\mathscr{M}^{i,t}_0} \phi^{i,t}_{T+1}\left(\int_{\mathscr{X}_{t+1}} U^{i,t}_{T+1}\left(h^i_t \cdot x\right) d\mathbb{P}^{i,t}_m(x)\right) d\pi^{i,t}(m)$$

$$s.t.\ h^i_t \cdot S_t = W^i_t \equiv h^i_{t-1} \cdot (S_t + D_t),$$

where

$$\mathbb{P}^{i,t}_m = t^{i,t}_T(m).$$

In particular, we have the possibility that the investor is characterized by

$$\left(I, U^{i,t}_{t+1}, \mathbb{P}^{i,t}_{\widehat{m}}\right),$$

i.e., the ambiguous non-risky case, where the prior distribution $\mathbb{P}^{i,t}_{\widehat{m}}$ is empirically determined on the basis of an econometric model that is selected such that arbitrage opportunities are excluded according to this model (and its estimate), given the prevailing prices in period t.[2] Arbitrage opportunities should be excluded according to the (estimated) econometric model, given the prevailing prices, to guarantee that the utility maximization does not yield unbounded results. Similarly to the "firms", the econometric model employed by and the preferences characterizing investor i might change over time, and are assumed to be unknown to the other economic agents.

In this benchmark economy there might be no risk, when the "firms" use deterministic rules, and the investors' preferences can be described by the ambiguous non-risky case. Price variations then occur solely due to ambiguity. Ambiguity arises due to the fact that the economic agents do not know each other's strategies to determine the dividends (the "firms") or the portfolios (the investors). Moreover, the exact way of intervention (the interaction between market makers and regulators) might be unknown. The economic agents only observe past data, consisting, for instance, of realized prices, trading volumes, etcetera. Different economic agents will have different preferences and endowments, but they also will use different econometric models. Together, this results in heterogeneity, causing trade and price fluctuations over time. Observing past data over a long period in order to recover the underlying deterministic system governing the economy will fail, due to time-varying preferences and rules for dividend payments, as well as changes in the econometric models employed over time.

To verify that the ambiguous non-risky variant of the benchmark model is coherent we might use the corollaries of the previous section. In case the regulators do not intervene, or only intervene by trading, the market makers will guarantee a temporary equilibrium, see, for instance, Grandmont (1988). In such a temporary equilibrium, the investors will be able to maximize their utility functions, subject to the budget constraint. It is not hard to see that the first order conditions of these constrained maximization problems will result in functions $m : \mathscr{X}_{T+1} \to \mathbb{R}$, one for each investor, which are strictly positive on the support of $\pi^{i,t}_{\mathscr{X}_{T+1}} \leftrightarrow \mathbb{P}^{i,t}_{\widehat{m}}$, and zero otherwise. In case the regulators inter-

[2] Alternatively, in case the investor would be a one-period ahead mean-variance investor, we can formulate the optimal investment problem as

$$\max_{h_T} \mathbb{E}_{\widehat{m}}(h_T \cdot X_{T+1}) - \frac{\gamma}{2} Var_{\widehat{m}}(h_T \cdot X_{T+1})$$

$$s.t.\ h_T \cdot S_T = W_T.$$

vene by imposing a price change limit (relating period $T-1$ and T), so that the market makers have to ration, the model might still be coherent, if a future zero price change (relating period T and $T+1$) combined with future zero dividend (in period $T+1$) is a possible future outcome, so that $\mathscr{X}_{T+1}$ includes the time T prices.

6 Conclusions

In this paper we investigated the relationship between ambiguity, no-arbitrage, and coherence, and we linked, in particular, an unambiguous risky view of the economy to an ambiguous non-risky view. Given an econometric model, we first discussed the various sources of ambiguity. Then we illustrated in the context of portfolio choice, making use of the recent literature, how ambiguity might be translated into preference orderings. Particularly, we linked the unambiguous risky and the ambiguous non-risky cases, both when dealing with expected utility and when dealing with mean-variance preferences. Then we introduced various ways of defining arbitrage opportunities, again focussing particularly on the distinction between the unambiguous risky case and the ambiguous non-risky case. We showed that absence of arbitrage opportunities in the ambiguous non-risky case corresponds to the concept of coherence, as introduced by Shafer and Vovk (2001). Finally, we introduced a benchmark artificial financial market, to show how an ambiguous non-risky (or, possibly ambiguous risky), but coherent economy can be modeled. The dynamics of such an economy can be investigated by combining the techniques of Shafer and Vovk (2001) and microscopic simulation techniques.

Using coherence as a guiding principle in constructing financial models instead of the assumption of no arbitrage opportunities broadens one's modeling possibilities considerably. In a companion paper Adriaens et al. (2006a) we further illustrate this.

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