Sufficient Conditions and Necessary Conditions for δ-stability

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Abstract

We provide sufficient conditions and necessary conditions for stability of an economy under structurally heterogeneous mixed recursive least squares/stochastic gradient learning of agents with possibly different degrees of inertia. We have found a unifying condition which is sufficient for convergence of an economy under such general type of adaptive learning towards rational expectations equilibrium for a broad class of economic models. We demonstrate and provide interpretation of this condition on an economic example.

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1 Introduction

Many papers are devoted to the question of stability. Mathematics studies the question of stability when studying systems of differential equations. Those mathematical results have been extensively used to study the behavior of dynamic economic models. The question of stability of a dynamic economic model is closely related to the question of the model's behavior around an equilibrium point and also to the question of equilibrium selection if we have multiple equilibria.

Until some time ago, stability issues were studied in models assuming rational expectations of agents. However, the need to study models under bounded rationality of agents was well argumented in Sargent (1993). Later this approach was also adopted (among others) in works of Evans and Honkapohja, and a standard argument in defense of bounded rationality can be found in Evans and Honkapohja (2001), as well as in Sargent (1993). The rational expectations (RE) approach implies that agents have a lot of knowledge about the economy (e.g., of the model structure and its parameter values). However, in empirical work economists, who assume RE equilibria in their theoretical model, do not know the parameter values and must estimate them econometrically. According to the argument of Sargent, it appears more natural to assume that agents in a given economy face the same limitations. It is then suggested to view agents as econometricians when forecasting the future state of the economy. Each time agents obtain new observations, they update their forecast rules. This approach introduces a specific form of bounded rationality captured by the concept of adaptive learning.

Besides, the bounded rationality approach can serve other purposes. For example, it can be used to test the validity of the RE hypothesis by checking if a given dynamic model converges over time to the RE equilibrium implied by the model (under RE hypothesis). Another role for the bounded rationality approach is that it can be used for equilibrium selection: some models have multiple equilibria under rational expectations, while the same models under bounded rationality do not because learning algorithms used by agents lead the model to select only one equilibrium.

Two possible algorithms that can be used to reflect bounded rationality of agents are generalized recursive least squares (RLS) and generalized stochastic gradient

(SG) algorithms[1]. Description of them can be found, for example, in Evans and Honkapohja (2001), Giannitsarou (2003), Evans, Honkapohja.and Williams (2005), Honkapohja and Mitra (2005). In fact, both algorithms are used by agents to update the estimates of the model parameters. RLS algorithm (non-generalized) can be obtained from OLS estimation of parameters, rewriting it in recursive form. Generalized RLS is derived from RLS by substituting gain sequence 1/t used in updating of regression coefficients with any decreasing gain sequence. Thus, generalized RLS algorithm has an equation for updating parameters that enter the model equations and also an equation for updating the second moments matrix. In contrast, (generalized) SG algorithm assumes the second moments matrix fixed[2].

At first, papers taking the bounded rationality approach of Sargent implied homogeneity among agents in the sense that all were assumed to follow the same updating algorithm (be it RLS or SG). One part of recent papers in the area of adaptive learning introduces heterogeneity in the updating procedure. For example, Giannitsarou (2003) introduces heterogeneous learning in order to check whether analysis of the model with heterogeneous learning is equivalent to the analysis of the same model under the hypothesis of a representative agent (implied by homogenous learning). Agents were assumed to be homogenous in all respects (including the structure of the equations they used for estimation) but the way they learned (updated the parameters of these equations). Honkapohja and Mitra (2005), on the other hand, introduce structural heterogeneity setup which includes types of heterogeneous learning considered by Giannitsarou (2003) as a special case. Both these papers study stability conditions of the economy in a given setup.

In both of these papers, agents differ not only in types of algorithms they used, but also in the relative weight they put on the updating term in the learning algorithm. First, an updating algorithm means that each period agents update the parameters of interest in the following way: the updated parameter estimate is equal to the previous estimate plus the most recent forecasting error (or some function of

[1] Both algorithms are examples of econometric learning. One more type of econometric learning is Bayesian learning. See Honkapohja and Mitra (2005) for references of other forms of learning – like bounded memory rules and non-econometric learning (including computational intelligence algorithms).

[2] A more detailed description of differences between the two algorithms can be found in Honkapohja and Mitra (2005).

it, in general) of the estimate multiplied by the gain parameter. The gain parameter, then, captures how important is the forecasting error for the agent. So, the gain sequence is said to reflect the degree of inertia of the agent in updating. See, e.g. Giannitsarou (2003) or Honkapohja and Mitra (2005) for more. Constant relative ratios between gain sequences in learning algorithms of any two different agent types constitute in the simplest case (as in Giannitsarou (2003)) relative degrees of inertia of updating.

In our paper we are solving the following open question posed by Honkapohja and Mitra (2005) — to find conditions for stability under structurally heterogeneous mixed RLS/SG learning with (possibly) different degrees of inertia. Though Honkapohja and Mitra (2005) have formulated a general criterion for such a stability and were able to solve for sufficient conditions for the case of a univariate model, they did not give an answer for conditions (necessary, and/or sufficient) on the structure of the model that would guarantee such a stability for the multivariate case with arbitrary number of agent types case.

As, in essence, the criterion for stability (in its sufficiency part) by Honkapohja and Mitra (2005) implies looking for sufficient conditions for D-stability of a particular stability Jacobian corresponding to the model, we use different sets of sufficient conditions for D-stability of this Jacobian and simplify them using particular structure of the model, trying to provide the derived conditions with some economic interpretation.

Specifically, in this paper we attempt to conduct a systematic analysis of the problem of deriving sufficient conditions for stability of the economy under structurally heterogeneous mixed recursive least squares/stochastic gradient (RLS/SG) learning for any (possibly different) degrees of inertia of agents. First, we analyzed what has been done so far in mathematics on deriving sufficient conditions for stability of a matrix in the most general setup of a matrix differential equation: $\dot{x} = Ax + b$, where A has the form $D\Omega$, whith D being a positive diagonal matrix. It has turned out that *the most general* results can be grouped according to the point from which the problem was approached. One group of results is based on Lyapunov theorem, and its application to D-stability by the theorem of Arrow and McManus; another group is based on the negative diagonal dominance condition which is sufficient for D-stability (McKenzie theorem); a third set of results can be derived from the characteristic equation analysis, using Routh-Hurwitz necessary and sufficient conditions for negativity of all eigenvalues of the polynomial of order n; and the last set of sufficient results in principle can be derived using an alternative definition of D-stability that allows to bypass the Routh–Hurwitz conditions.

Among the approaches mentioned above the ones that are based on negative diagonal dominance, characteristic equation analysis, and the alternative definition (criterion) of D–stability turn out to be fruitful, each to different extent. (The condition based on Lyapunov theorem looks very theoretical and economically intractable in our case.) The negative diagonal dominance and the alternative definition of D-stability give us the "aggregate economy stability" and the "equal sign" sufficient conditions. As for the characteristic equation analysis, we have been able to derive a block of necessary conditions using the negativity of eigenvalues requirement, bypassing the Routh-Hurwitz conditions since they are quite complicated and do not have economic interpretation. Each group of the results has been studied in application to the particular setup of models we are working with in order to make the procedure testing for stability more tractable and at the same time to attach some economic interpretation to this very procedure.

2 The Setup

Deriving conditions for stability of the economy under structurally heterogeneous mixed RLS/SG for any (possibly different) degrees of inertia of agents, we naturally employ the general framework and notation from Honkapohja and Mitra (2005), who were first to formulate general criterion for stability of the economy under mixed RLS/SG heterogeneous learning. "Structurally heterogeneous" here refers to structural heterogeneity, which means that expectations and learning rules of different agents are different, as well as may be different their fundamental characteristic, such as preferences, endowments, and technology (as opposed to structural homogeneity, which corresponds the assumption of a representative agent).

Structural heterogeneity in the setup of Honkapohja and Mitra (2005) is expressed through matrices A_h, which are assumed to incorporate the mass ζ_h of each agent type. So, $A_h = \zeta_h \cdot \hat{A}_h$, where $\hat{A}_h$ is defined as describing how agents of type

h respond to their forecasts. So these are the structural parameters characterizing a given economy. Those may be basic characteristics of agents, like those describing their preferences, endowments, and technology. Structural heterogeneity means that all $\hat{A}_h$'s are different for different types of agents. When $\hat{A}_h = A$, the economy is structurally homogenous.

"Mixed RLS/SG learning" refers to persistently heterogeneous learning, defined by Honkapohja and Mitra (2005) as the one arising when different agents use different types of learning algorithms. In the setup of H&M these are RLS and SG algorithms.

More on this (as well as some useful reference for more detailed study of the terms) can be found in Honkapohja and Mitra (2005) (In order not to repeat Honkapohja and Mitra (2005), we just briefly present the general setup and general criterion of stability result. For full presentation of RLS/SG learning and setup please see Honkapohja and Mitra(2005))

The class of linear structurally heterogeneous models with S types of agents with different forecasts is presented by

$$y_t = \alpha + \sum_{h=1}^{S} A_i \hat{E}_t^h y_{t+1} + B w_t,$$
$$w_t = F w_{t-1} + v_t,$$

where y_t is $n \times 1$ vector of endogenous variables, w_t is $k \times 1$ vector of exogenous variables, v_t is a vector of white noise shocks, $\hat{E}_t^h y_{t+1}$ are (in general, non-rational) expectations of the endogenous variable by agent i, $M_w = \lim_{t\to\infty} w_t w_t'$ — is positive definite, F ($k \times k$ matrix) is such that w_t follows stationary VAR(1) process

The vector form presented above is a reduced form of the model describing the whole economy, i.e. it is an equation corresponding to the inter–temporal equilibrium of the dynamic model. Expectations (in general, not rational) of different agent types influence the current values of endogenous variables.

We also stress the "diagonal" environment (the reason for which will be given below) which we analyze, namely $F = diag(\rho_1, ..., \rho_k)$, $M_w = diag\left(\frac{\sigma_1^2}{1-\rho_1^2}, ..., \frac{\sigma_k^2}{1-\rho_k^2}\right)$.

In forming their expectations about next period endogenous variable, agents are assumed to believe that economic system develops according to the following model, which is called agents' perceived law of motion (PLM).

$$y_t = a_{i,t} + b_{i,t} w_t.$$

Mixed learning of agents is introduced as follows. Part of agents, $i = 1, ..., S_0$, is assumed to use RLS learning algorithm while others, $j = S_0 + 1, ..., S$, are assumed to use SG learning algorithm. Moreover, all of them are assumed to use possibly different degrees of responsiveness to the updating function that are presented by different degrees of inertia δ_h, which, in formulation of Giannitsarou (2003), are constant coefficients before common for all agents deterministic decreasing gain sequence in learning algorithm. (Honkapohja and Mitra use more general formulation of degrees of inertia as constant limits in time of expected ratios of agents' random gain sequences and common for all deterministic decreasing gain sequence satisfying certain regularity conditions.)

After denoting $z_t = (1, w_t)$ and $\Phi_{i,t} = (a_{i,t}, b_{i,t})$, the formal presentation of learning algoritms in this model can be written as follows.

RLS: for $i = 1, .., S_0$

$\Phi_{i,t+1} = \Phi_{i,t} + \alpha_{i,t+1} R_{i,t}^{-1} z_t \left(y_t - \Phi_{i,t}' z_t\right)'$

$R_{i,t+1} = R_{i,t} + \alpha_{i,t+1} \left(z_{t-1} z_{t-1}' - R_{i,t}\right)$

SG: for $j = S_0 + 1, ..., S$

$\Phi_{j,t+1} = \Phi_{j,t} + \alpha_{j,t+1} z_t \left(y_t - \Phi_{j,t}' z_t\right)'$

Honkapohja and Mitra show that stability of REE, Φ_t, in this model is determined by stability of the ODE:

$\frac{d\Phi i}{d\tau} = \delta_i \left(T(\Phi')' - \Phi_i\right), \; i = 1, ..., S_0$

$\frac{d\Phi j}{d\tau} = \delta_j M_z \left(T(\Phi')' - \Phi_j\right), \; j = S_0 + 1, ..., S,$

where $M_z = \lim_{t\to\infty} E z_t z_t'$.

The conditions of stability of this ODE give the general criterion of stability result for this class of models presented in Proposition 5 in Honkapohja and Mitra (2005)

In the economy above mixed RLS/SG learning converges globally (almost surely) to the minimal state variable (MSV)[3] solution if and only if the matrices $D_1\Omega$ and $D_w\Omega_F$ have eigenvalues with negative real parts, where

[3] As is mentioned in ch. 8 of Evans and Hokapohja (2001), the concept of the MSV solution was introduced by McCallum (1983) for linear rational expectations models. As is defined in E&H(2001), this is the solution that depends linearly on a set of variables (in our case it is the vector of exogenous variables and the intercept); this solution is such that there is no other solution that depends linearly on a smaller set of variables.

$$D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix}, \Omega = \begin{pmatrix} A_1 - I_n & \cdots & A_S \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_S - I_n \end{pmatrix}$$

$$D_w = \begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix}, \Omega_F = \begin{pmatrix} F' \otimes A_1 - I_{nk} & \cdots & F' \otimes A_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 & \cdots & F' \otimes A_S - I_{nk} \end{pmatrix},$$

$D_{wi} = \delta_i I_{nk}, i = 1, ..., S_0,$ $D_{wj} = \delta_j \left(M_w \otimes I_n \right), j = S_0 + 1, ..., S.$

In the "diagonal" environment we consider, the problem of finding stability conditions of both $D_1\Omega$ and $D_w\Omega_F$ is simplified to finding stability conditions of $D_1\Omega$ and $D_1\Omega_{\rho_i}$,where Ω_{ρ_i} is obtained from Ω by substituting all A_h with $\rho_i A_h$, where $|\rho_i| < 1$ as w_t follows stationary VAR(1) process by setup of the model. We also use special blocked—diagonal structure of the matrix D_1 which is the feature of the dynamic environment in this class of models.(In a sense these positive diagonal D-matrices now may be called positive blocked—diagonal δ-matices.) It allows us to formulate the concept of δ–stability by analogy to the terminology of the concept of D–stability, studied for example in Johnson (1974).

Definition 1 *Given n, the number of endogenous variables, and S, the number of agent types, δ-stability is defined as stability of the economy under structurally heterogeneous mixed RLS/SG learning for any (possibly different) degrees of inertia of agents, $\delta > 0$.*

$\delta-$stability, thus formulated, has the same meaning in models with heterogeneous learning described above as has $E-$stability condition in models with homogeneous RLS learning. $E-$stability condition is the condition for asymptotic stability of an REE under homogeneous RLS learning. The REE of the model is stable if it is locally asymptotically stable under the following ODE:

$\frac{d\theta}{d\tau} = T(\theta) - \theta,$

where θ are the estimated parameters from agents PLMs, $T(\theta)$ is a mapping of PLM parameters into parameters of actual law of motion (ALM), which is obtained when we plug forecasts functions based on agents PLMs into the reduced form of the model and τ is a "notional" ("artificial") time. The fixed point of this ODE is the REE of the model.

In what follows we consider conditions of thus formulated $\delta-$stability, derived using different approaches discussed above. Section 2 is devoted to sufficient conditions of δ-stability, among which are aggregated economy and "equal sign" conditions. In Section 3 we present necessary conditions of $\delta-$stability that are based on characteristic equation. And in Section 4 we show how sufficient aggregated economy conditions can be interpreted on an economic example, the model of simultaneous markets with structural heterogeneity.

3 Sufficient conditions

3.1 Aggregated economy conditions

The **negative diagonal dominance** approach allows one to show that in the setting specified above δ–stability depends on E–stability of the aggregated economy which is the upper boundary of aggregated economies with weights of aggregation across agents, ϕ, and weights of aggregation across endogenous variables, ψ.

We have been encouraged by the result that follows from Propositions 2 and 3 in Honkapohja and Mitra (2005) that for stability under heterogeneous RLS or SG learning with the same degrees of inertia, stability in the economy aggregated across agent types (average economy) tuns out to be crucial. Following Honkapohja and Mitra (2005), who aggregated across agents by introducing the concept of average (aggregated across agents) economy, we also are looking for the concept of aggregated economy that has to be crucial for stability under structural mixed RLS/SG heterogeneous learning with different degrees of inertia. ($\delta-$stability)

The basic idea is that there has to be a way of how to aggregate an economy in an economically reasonable way, so that stability in the aggregated economy is sufficient for $\delta-$stability. Another idea is to circumvent the problem of looking for weights of aggregation that guarantee stability (which follows from the negative diagonal dominance requirement) by finding such kind of a unifying economy coefficient - the coefficient before expectations in the aggregated economy (we will call it aggregated β–coefficient) - that will work as an indicator of stability: if it is less than one, then there exists an economically reasonable aggregated economy, which implies $\delta-$stability.

It has also turned out that it is possible to find an upper boundary for economic models with the same absolute values of coefficients (elements of matrices A), but with possibly different signs, such that E–stability of such an upper boundary aggregated economy implies E–stability in all aggregated economies and the δ–stability in all set of the models with the same absolute values of coefficients. This finding together with the unifying aggregated β–coefficient allows us to see how robust the model is to possible change of signs of coefficients, and to see how robust is $\delta-$stability in this economy to a change in economic parameters. As an example, the policy maker may know some structural coefficients in the economy and have to choose some parameters itself (like the ones for the policy rule). This formula allows it to see what is the range of the parameters it may choose in order to make sure that the economy will be δ–stable. Of course, it may include into consideration the possible range in other agents' coefficients that it has estimated but knows that they belong to some interval with some probability (situation typical for statistical interval estimation).

We proceed with aggregation of the economy starting from the following aggregation across agents used by Honkapohja and Mitra (2005): $y_t = \alpha + A^M \hat{E}_t^{AV} y_{t+1} + B w_t$. It turns out that it is convenient, in addition to the aggregation across agents (considered by Honkapohja and Mitra (2005)), to consider aggregation across endogenous variables. The economy aggregated across endogenous variables will no longer be a vector but a scalar, which means that it can characterize many economies. This aggregation across endogenous variables can be interpreted as the construction of a limiting hyperplane on the space of vectors of endogenous variables — such hyperplane that all vectors in the subspace limited by this hyperplane are stable economies. It is like $ax + by \leq c$ (if we consider economies characterized by a two-dimensional vector of endogenous variables (x, y)): all vectors (x, y) satisfying the constraint are stable if the limiting aggregated economy characterized by c is stable.

We rewrite the formulas used by Honkapohja and Mitra (2005) for average expectations as

$$E_t^{AV} y_{t+1} = (A^M)^{-1} \left(S \sum_{h=1}^{S} \tfrac{1}{S} A_h E_t^h y_{t+1} \right)$$

$$A_M = S \sum_{h=1}^{S} \tfrac{1}{S} A_h = S \sum_{h=1}^{S} \tfrac{1}{S} \zeta_h \hat{A}_h$$

So this can be interpreted as if we take the weight of each agent type in calculating aggregate expectations of one representative agent to be equal to $\frac{1}{S}$ and then multiply it by S in order to be consistent with the model in which we have S types of agents (in the sense that we do not shrink the size of the economy to one representative agent but preserve its size by replacing each type of agent by a representative agent). In general, when aggregating expectations one may use different weights for different types of agents summed up to one in order to reflect the relative importance of a particular agent type expectations in the aggregated economy. So, we first create a representative agent type by averaging across all agent types (assigning a weight to each type and summing over all types), then we aggregate over all types by multiplying the representative (average) agent type by S in order to preserve the size of the aggregated economy.

Hokapohja and Mitra (2005) use as weights in the formula for the average (aggregate) economy the mass of each agent type ζ_h. The mass of each agent type was introduced above in section "The Setup", and we also said that this mass is incorporated in matrices A_h. We introduce additional dimension to weighting the agent types.(This dimension can be interpreted as expressed via equal weights $\frac{1}{S}$ in Honkapohja and Mitra (2005)). We can interpret it as follows. We can assume that the share of each agent type expectations in the average expectations of the population is determined not only by their mass in the population (their physical share), but also by each type's influence, other than their share in the population (e.g. political power or other type of influence in the social life of the whole population; it can be, for example, the influence mass media have on the minds of the rest of the population, just like advertising of some product can influence the demand for this product). Even a group of agents that does not have a large share in the population can have significant influence on the overall expectations. We can, of course model this second dimension in expectations of separate agent types by making the expectations of each agent type a function of expectations of other agent types. For example, model how the type of agents called "mass media" influence some other agent types. This would reflect reality more adequately, but will be very tricky to model (too many interrelations), leave alone to solve such kind of model. Our interpretation of the additional weights as the influence of each group on the overall expectations in the economy can be viewed as a shortcut: even if some agent types influence other agent types, in the end this will lead to influence the overall expectations in the economy, so by assigning additional weights to each agent type we provide a measure of the share of influence of each agent type in the overall expectations, bypassing the intermediate step of measuring the influence of each agent type on other agent types separately.

If we write the aggregated economy using different weights for agents, we will get

$$\hat{E}_t^{Weighted} y_{t+1} = \left(A^{Weigted}\right)^{-1} \left(S \sum_{h=1}^{S} \phi_h A_h \hat{E}_t^h y_{t+1} \right)$$

$$A^{Weigted} = S \sum_{h=1}^{S} \phi_h A_h = S \sum_{h=1}^{S} \tfrac{1}{S} \phi_h \zeta_h \hat{A}_h,$$

where $\phi_h > 0$ are weights of single agent types used in calculating aggregate expectations, such that $\sum_{h=1}^{S} \phi_h = 1$. Later we discuss which weights should be used to reflect the relative weight of each agent type in aggregation of expectations.

Next, given the weights of aggregation across endogenous variables $\psi_i > 0$, $\sum_{i=1}^{n} \psi_i = 1$, and across agents $\phi_h > 0$, $\sum_{h=1}^{S} \phi_h = 1$, we aggregate the economy in the following way

$$\begin{aligned}
&\psi_1 y_{1t} + ... + \psi_n y_{nt} = \psi_1 \alpha_1 + ... + \psi_n \alpha_n + \\
&+ S\phi_1 \sum_i \psi_i a_{i1}^1 \hat{E}^1 y_{1t+1} + ... + S\phi_S \sum_i \psi_i a_{i1}^S \hat{E}^S y_{1t+1} + ... + \\
&+ S\phi_1 \sum_i \psi_i a_{in}^1 \hat{E}^1 y_{nt+1} + ... + S\phi_S \sum_i \psi_i a_{in}^S \hat{E}^S y_{nt+1} = \\
&= \psi_1 \alpha_1 + ... + \psi_n \alpha_n + \\
&+ \sum_{h=1}^{S} S\phi_h \underbrace{\sum_i \psi_i \sum_j a_{ij}^h}_{\beta_h} \frac{\sum_i \psi_i \sum_j a_{ij}^h \left(\hat{E}^h y_{jt+1}\right)}{\sum_i \psi_i \sum_j a_{ij}^h} = \\
&= \psi_1 \alpha_1 + ... + \psi_n \alpha_n + \beta_{aggreg}(\psi, \phi) E_{aggreg}(\psi_1 y_{1t+1} + ... + \psi_n y_{nt+1}), \text{ where} \\
&\beta_{aggreg}(\psi, \phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j a_{ij}^h, \\
&E_{aggreg}(\psi_1 y_{1t+1} + ... + \psi_n y_{nt+1}) = \\
&= \left(\sum_{h=1}^{S} S\phi_h \underbrace{\sum_i \psi_i \sum_j a_{ij}^h}_{\beta_h} \right)^{-1} \times \sum_{h=1}^{S} S\phi_h \underbrace{\sum_i \psi_i \sum_j a_{ij}^h}_{\beta_h} \frac{\sum_i \psi_i \sum_j a_{ij}^1 \left(\hat{E}^h y_{jt+1}\right)}{\sum_i \psi_i \sum_j a_{ij}^h}.^{4}
\end{aligned}$$

[4] We assumed that these weights are such that we may divide over the corresponding coefficient

It is also useful to consider the economy that bounds above all the possible economies with different signs of a_{ij}^h aggregated using weights ψ. This is obviously the model which is written in absolute values. (When all elements in the model, a_{ij}^h, endogenous variables and their expectations are positive, this limiting model exactly coincides with the model considered (So, this is attainable supremum)). Thus we have the following limiting aggregated model:

$\psi_1 y_{1t} + ... + \psi_n y_{nt} \leq \psi_1 |y_{1t}| + ... + \psi_n |y_{nt}| \leq$

$\leq \psi_1 |\alpha_1| + ... + \psi_n |\alpha_n| + \beta_{aggreg}^{\text{mod}}(\psi, \phi) E_{aggreg}^{\text{mod}}(\psi_1 |y_{1t+1}| + ... + \psi_n |y_{nt+1}|)$, where

$\beta_{aggreg}^{\text{mod}}(\psi, \phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j |a_{ij}^h|$

If this limiting economy is E–stable (i.e. $\beta_{aggreg}^{\text{mod}}(\psi, \phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j |a_{ij}^h| < 1$), then all the corresponding aggregated economies with various combinations of signs of a_{ij}^h are E–stable ($\beta_{aggreg}(\psi, \phi) = S \sum_h \phi_h \sum_i \psi_i \sum_j a_{ij}^h < 1$).

The structure of this limiting aggregate coefficient is as follows.

$\sum_i \psi_i |a_{ij}^h|$ is the coefficient before expectation of endogenous variable j in the aggregated economy composed of one single agent type h. Notice that this coefficient is calculated for expectation of endogenous variable j, that enters the aggregated product with coefficient ψ_j. So, we may name the ratio $\frac{\sum_i \psi_i |a_{ij}^h|}{\psi_j}$ the endogenous variable's j "own" expectations relative coefficient. By looking at the values of these coefficients we will be able to judge about the weight a particular agent type has in the economy in terms of the aggregated β–coefficient. The next proposition is formulated in terms of these relative coefficients and stresses the fact that weights of agents in calculating aggregated expectations have to be put into accordance with this economic intuition in order to have stability under heterogeneous learning.

Proposition 2 *If there exists at least one pair of vectors of weights for aggregation of endogenous variables ψ and weights ϕ for aggregation of agents such that for each agent every endogenous variable's "own" expectations relative coefficient is less than the weight of the agent used in calculating aggregated expectations (i.e. weights of*

consisted weighted sums of $a_{ij}'s$. In case, when all a_{ij}, participating in coefficient over which we have to divide are zero, this means that we simply do not divide over it, since these expectations simply do not enter the aggregation,as summing up to zero.

agents are put into accordance with their endogenous variable's "own" expectations relative coefficients), then the economy is $\delta-$stable for any n and S.

Proof: see Appendix

But this proposition above does not give a real rule of thumb (as it implies looking for systems of weights) to say that a particular system is stable under heterogeneous learning. For this purpose we have constructed four maximal aggregated β–coefficients described below. If they are less then one the economic system is $\delta-$stable.

Now, we go even further looking for an upper boundary by considering not only any signs of a_{ij}, but also arbitrary values of weights ψ and ϕ. It is clear that any aggregated economy with any weights will be bounded above by the following maximal aggregating economy.

$\psi_1 y_{1t} + \psi_2 y_{2t} + ... + \psi_n y_{nt} \leq$

$\leq |y^{aggreg}_{\max}| = \psi_1 |\alpha_1| + ... + \psi_n |\alpha_n| + \beta^1_{\max} E^{\max}_{aggreg} (\psi_1 |y_{1t+1}| + ... + \psi_n |y_{nt+1}|)$, where $\beta^1_{\max} = S \sum\limits_j \max\limits_{h,i} |a^h_{ij}|$.

It is possible to derive other upper boundaries for subsets of aggregated economies with either equal weights of agents $\frac{1}{S}$ or equal weights of endogenous variables $\frac{1}{n}$, or both.

In case of equal weights of agents $\frac{1}{S}$ and arbitrary weights of endogenous variables ψ, we have

$|y^{aggreg}_{\max}| = \psi_1 |\alpha_1| + ... + \psi_n |\alpha_n| + \beta^2_{\max} E^{\max}_{aggreg} (\psi_1 |y_{1t+1}| + ... + \psi_n |y_{nt+1}|)$, where $\beta^2_{\max} = \max\limits_i \sum\limits_h \sum\limits_j |a^h_{ij}|$.

In case of equal weights of endogenous variables $\frac{1}{n}$ and arbitrary weights of agents ϕ, we have

$|y^{aggreg}_{\max}| = \psi_1 |\alpha_1| + ... + \psi_n |\alpha_n| + \beta^3_{\max} E^{\max}_{aggreg} \left(\frac{1}{n} |y_{1t+1}| + ... + \frac{1}{n} |y_{nt+1}|\right)$, where $\beta^3_{\max} = S \sum\limits_i \max\limits_{h,j} |a^h_{ij}|$.

In case of equal weights of agents $\frac{1}{S}$ and equal weights of endogenous variables $\frac{1}{n}$, we have

$|y^{aggreg}_{\max}| = \psi_1 |\alpha_1| + ... + \psi_n |\alpha_n| + \beta^4_{\max} E^{\max}_{aggreg} \left(\frac{1}{n} |y_{1t+1}| + ... + \frac{1}{n} |y_{nt+1}|\right)$, where $\beta^4_{\max} = \sum\limits_h \max\limits_j \left\{ \sum\limits_i |a^h_{ij}| \right\}$.

Notice that each new boundary is constructed in such a way that it does not replicate the boundary for a broader set of aggregated models to which this particular model belongs. It is possible to do by applying the max operator to different grouping of elements of sum that becomes possible, for particular subset of aggregated models and which was not possible to apply for broader set. Under equal $\left|a_{ij}^h\right| = |a|$ all these maximal β–coefficients coincide with $\beta_{aggreg}^{\text{mod}}(\psi, \phi) = nS\,|a|$. So, these are attainable maxima.

So, we have managed to aggregate the economy into one dimension and to find the limiting aggregated economies that bound all of such aggregated economies within a particular subset. If one of these limiting aggregated economies is E–stable (i.e. if at least one of the maximal aggregated $\beta-$ coefficients is less than one), then all aggregated subeconomies from a particular subset are E–stable.

Now we are ready to formulate the result which stresses the key role of E–stability in the aggregated economy on the stability under structural heterogeneous learning (recall Proposition 2 and Proposition 3 in Honkapohja and Mitra (2005)).

The key result is as follows.

Proposition 3 *If one of the limiting aggregated economies is E–stable (i.e. one of maximal aggregated β–coefficients is less than one), then the economy is $\delta-$stable for any n and S. (Notice that the aggregated whole economy,aggregated single economies and all aggregated subeconomies are also stable under this condition)*[5].

Proof: see Appendix

This result gives the direct rule how to construct $\delta-$stable economies

We think that this is quite a strong result that says that there is one economic unifying condition (such as aggregated β–coefficient less than one) such that when it holds true all the economies with the same absolute values, but all possible various combinations of signs of a_{ij}^h are stable under heterogenous learning with mixed RLS/SG learning with any different positive degrees of inertia. It can be very useful in the case when one does not know the exact sign of some coefficient in matrix h,

[5] Note that the economies mentioned in this comment in brackets are economies aggregated without taking absolute values, so not the limiting economy. Then, the aggregated whole economy means that the aggregation is done across all agent types, while an aggregated subeconomy means that the aggregation is partial, i.e. that we sum over a part of agent types, and an aggregated single economy is a particular case of a subeconomy with only one type of agent.

but may estimate that its absolute value belongs to some interval.(It is common situation in policy making). Moreover, these coefficients may change sign in the economy during the time. So, this condition is to show how robust stability of the model to a change of sign of some coefficients. Fixing some known coefficients in these aggregated β–coefficients, we may see how the value of other coefficients is flexible for the economy to remain stable under structural heterogeneous learning with mixed RLS/SG learning when it is possible that economic agents have different degrees of inertia and from time to time change them.

Implications for more simple cases

From the condition in the proposition above, we get that for the case $n = 1$ and for structurally homogenous case the following propositions hold true.

Proposition 4 *The univariate economy is stable under all forms of heterogeneous learning we considered for any combination of signs of coefficients if and only if* $|A_1| + |A_2| + ... + |A_s| < 1.$

Proof: Obvious: necessary condition for Ω to be stable under any delta is $A_1 + ... + As < 1$ (From the condition on the determinant of the $-\Omega$ which has to be positive This determinant equals $-(A_1 + ... + As) + 1)$). For the above condition to hold true for any signs of A_h it is necessary and sufficient that $|A_1|+|A_2|+...+|A_s| < 1$.

The same condition implies that all subeconomies are stable under any form of structural heterogeneous learning.

Proposition 5 *For structurally homogeneous economy:* $A_h = \zeta_h A$, $\zeta_h > 0, \sum_{h=1}^{S} \zeta_h = 1$,*to be* δ*–stable it is sufficient that at least one of the following limiting aggregated* β*–coefficients:* $\max_i \sum_j |a_{ij}|$ *and* $\max_j \sum_i |a_{ij}|$ *is less than one.*

Proof: direct application of Proposition 2

3.2 "Equal sign" conditions

Following the steps of the proof of observation (iv) in Johnson (the formulation of this observation is presented in Appendix A), which is in fact alternative definition of $D-$stability, we get an alternative definition of blocked—diagonal $(D_b)-$stability, that is stability of $D_b\Omega$ for any positive blocked—diagonal matrix D_b. This alternative definition of D_b-stability is then used to derive conditions for $\delta-$stability.

Definition 6 *(D_b-stability) Matrix A of size $nS \times nS$ is D_b-stable if D_bA is stable for any positive blocked–diagonal matrix $D_b = diag(\delta_1, ..., \delta_1, ..., \delta_S, ..., \delta_S)$.*

Proposition 7 *(Alternative definition of D_b-stability). Consider $M_{nS}(C)$, the set of all complex $nS \times nS$ matrices, D_{bnS}, the set of all $nS \times nS$ blocked–diagonal matrices with positive diagonal entries. Take $A \in M_{nS}(C)$ and suppose that there is an $F \in D_{nS}$ such that FA is stable. Then A is D_b–stable if and only if $A \pm iD_b$ is non–singular for all $D_b \in D_{bnS}$. If $A \in M_{nS}(R)$, – the set of all $nS \times nS$ real matrices, then "$\pm$" in the above condition may be replaced with "$+$" since, for a real matrix, any complex eigenvalues come in conjugate pairs.*

Proof. (The proof is just a modification of the proof of observation (iv) in Johnson for our blocked–diagonal case)

Necessity. Let A be D_b-stable, that is EA is stable for all positive blocked–diagonal $E \in D_{bnS}$. This means that $\pm i$ cannot be an eigenvalue of matrix EA for any $E \in D_{bnS}$. That is $EA \pm iI$ is non–singular for all $E \in D_{bnS}$, or $A \pm iD_b$ is non–singular for all $D_b = E^{-1} \in D_{bnS}$.

Sufficiency. By contradiction, let A be not D_b–stable. Thus, we have that there exists some $E \in D_{bnS}$ such that FA is stable, while EFA is not stable. By continuity, it follows that either value, $\pm i$, is an eigenvalue of $\frac{1}{\alpha}\left(tE + (1-t)I\right)FA$ for some $0 < t \leq 1$ and $\alpha > 0$. So, $A \pm iD_b$ is singular for $D_b = \alpha\left(tE + (1-t)\,I\right)^{-1} \in D_{bnS}$. Contradiction. ■

Taking F as an identity matrix, and D as $diag(\frac{1}{\delta_1}, ..., \frac{1}{\delta_1}, ..., \frac{1}{\delta_S}, ..., \frac{1}{\delta_S})$ in the above proposition, we get the following necessary and sufficient condition for $\delta-$stability of Ω:

$\Omega-$stable and

$$\det\left[\frac{-A_1}{1+\frac{i}{\delta_1}} + ... + \frac{-A_S}{1+\frac{i}{\delta_S}} + I\right] =$$

$$= \det\left[\left(\frac{1}{1+\frac{1}{\delta_1^2}}(-A_1) + ... + \frac{1}{1+\frac{1}{\delta_S^2}}(-A_S) + I\right) \pm i\left(\frac{\frac{1}{\delta_1}}{1+\frac{1}{\delta_1^2}}(-A_1) + ... + \frac{\frac{1}{\delta_S}}{1+\frac{1}{\delta_S^2}}(-A_S)\right)\right] \neq 0, \forall\delta > 0$$

For case $n = 1$ it simplifies to

$\Omega-$stable and
$\left(\frac{1}{1+\frac{1}{\delta_1^2}}(-A_1) + ... + \frac{1}{1+\frac{1}{\delta_S^2}}(-A_S) + 1\right) \neq 0,$
or $\left(\frac{\frac{1}{\delta_1}}{1+\frac{1}{\delta_1^2}}(-A_1) + ... + \frac{\frac{1}{\delta_S}}{1+\frac{1}{\delta_S^2}}(-A_S)\right) \neq 0,$
or both.

The **alternative definition of D–stability** approach allows us to derive the "equal sign" conditions for the cases $n = 1, 2$ and necessary and sufficient conditions for δ–stability for $n = 1$.

Proposition 8 *(Criterion of $\delta-$stability in the univariate case) In case $n = 1$ the Ω is $\delta-$stable if and only if Ω is stable and at least one of the following holds true: equal sign condition (all A_i are greater or equal than zero and at least one is strictly greater than zero or all A_i are less or equal than zero and at least one is strictly less than zero), or all of subeconomies are not unstable and at least one of them in each size is stable.*

Proposition 9 *In case $n = 2$, Ω is $\delta-$stable if Ω is stable and an "equal sign" condition holds true, where the "equal sign" condition looks in general case as follows*

$\det(-A_i) > 0, [\det mix(-A_i, -A_j) + \det mix(-A_j, -A_i)] > 0, i \neq j, M_1(-A_i) > 0,$

or

$\det(-A_i) < 0, [\det mix(-A_i, -A_j) + \det mix(-A_j, -A_i)] < 0, i \neq j, M_1(-A_i) < 0.$

(Here we introduce the concept of a pairwise mixed economy which is an economy characterized by a matrix of structural parameters composed by mixing columns of a pair of matrices A_i, A_j, for any i, $j = 1, ..., S$.)

The condition above may follow from the following (more understandable) sufficient condition:

Proposition 10 *In case $n = 2$, Ω is $\delta-$stable if Ω is stable and all eigenvalues of A_i (matrices of coefficients of single economies) and of pairwise mixes (matrices of coefficients of pairwise mixed economies) are negative or all eigenvalues of A_i and of pairwise mixes are positive.*

Unfortunately, though similar "equal sign" conditions naturally follow from the alternative definition of D–stability for cases $n > 2$, stability of Ω and a similar "equal sign" condition are not sufficient for $\delta-$stability in this case.

For example, a similar "equal sign" condition for case $n = 3$ looks like

$M_3 (mix(-A_i, -A_j, A_k)) > 0$, $M_2(mix(-A_i, -A_j)) > 0$, $M_1(-A_i) > 0$ or

$M_3 (mix(-A_i, -A_j, A_k)) < 0$, $M_2(mix(-A_i, -A_j)) < 0$, $M_1(-A_i) < 0$

Here, the $M_n(mix())$ operator means the sum of all possible principal minors of size n of a particular mix between matrices.

4 Necessary conditions

The **characteristic equation** approach (that in our formulation leaves aside the intractable Routh-Hurwitz conditions) allows one to derive strong necessary conditions for δ–stability, that provide an easy test for non–$\delta-$stability of the model. (Note that necessary conditions do not require diagonal structure of F and M_w.)

Condition (*) All sums of the same-size principal minors of $diag(\delta_r)(-\Omega_r)$ are nonnegative (where $(-\Omega_r)$ corresponds to a subeconomy of the economy under consideration) for all subeconomies r for all positive block-diagonal $diag(\delta_r)$.

Proposition 11 *Necessary condition for $\delta-$stability: For economy to be δ-stable it is necessary that condition (*) holds true.*

The condition above can not be used as a test for non-$\delta-$stability, as it requires checking all subeconomies sums of minors for all $\delta_r > 0$.

That is why below we have constructed the condition that has the direct testing application.

Proposition 12 *Necessary condition for $\delta-$stability: For an economy to be $\delta-$stable, it is necessary that all sums of the same-size principal minors of minus matrices corresponding to subeconomies are non-negative for each subeconomy, $i_1, ..., i_p$.*

We think that this is quite a strong necessary condition, which implies that a lot of models will not satisfy it, and will not be $\delta-$stable. Note that stability of each single economy and subeconomies is a sufficient condition for the condition above to hold true. A weaker requirement that all subeconomies are not unstable (non-positive real parts of eigenvalues) is also sufficient.

5 Economic Example

We demonstrate the aggregate economy suffcient conditions on the model of simultaneous markets with structural heterogeneity[6]

The economic environment is given by the following equations

$p_t = l + vd_t + \varepsilon_t$ is demand function in matrix form for different goods $j = 1, ..., J$

p_t is a $J \times 1$ vector of prices(which are endogenous variables in this model), l is a vector of intercepts, v is a $J \times J$ matrix, which corresponds to the inverse of the matrix of price effects. $d(t)$ is a vector of quantities of the J goods, $\varepsilon_{j,t} = f_j \varepsilon_{j,t-1} + v_{j,t}$, $\varepsilon_{j,t}$ are demand shocks, $|f_j| < 1$, $v_{j,t}$ are independent white noises.

There are S types of suppliers with supply functions:

$s_t^h = g^h + n^h \hat{E}_{t-1}^h p_t$, $h = 1, ..., S$,

which depend on expected price due to a production lag. Each supplier produces all J goods. $s(h, t)$ is a $J \times 1$ vector of goods supplied by type h supplier.

It is further assumed that different outputs are produced in independent processes by each producer h, so n^h is a positive diagonal matrix. Expectations (non-rational, in general) of prices are formed by each supplier at the end of period $t - 1$ before realization of demand shock $\varepsilon_{t.}$

Market clearing condition,$d_t = \sum_{h=1}^{S} s_t^h$, leads to the following reduced form

$$p_t = l + v \left(\sum_{i=1}^{S} g^i \right) + \sum_{h=1}^{S} vn^h \hat{E}_{t-1}^h p_t + \varepsilon_t$$

For the case with equal weights of single agent types used in calculating aggregate expectations, the aggregated stability sufficient condition for this model has the form:

$$\sum_i \psi_i |v_{ij}| < \frac{\psi_j}{Sn_{jj}^h}, \forall j, h.$$

This condition can be derived by the direct application of Proposition 2 to the given model.

It is clear that the above condition follows from $\sum_i \psi_i |v_{ij}| < \frac{\psi_j}{S \max_h \{ n_{jj}^h \}} \leq \frac{\psi_j}{Sn_{jj}^h}, \forall j, h$, which is to say that if $\sum_i \psi_i |v_{ij}| < \frac{\psi_j}{S \max_h \{ n_{jj}^h \}}, \forall j$ holds, then the condition implied by Proposition 2 holds as well.

[6]The authors express sincere thanks to Seppo Honkapohja who suggested to use this example.

And the last condition is condition for stability of the aggregated cobweb model in terms of the inequality relation between slopes of the demand and supply curves:

The demand curve for the price index[7]

$$P_t = \sum_i \psi_i p_{it} = \left(\sum_i \psi_i v_{i1}\right) d_{1t} + ... + \left(\sum_i \psi_i v_{iJ}\right) d_{Jt} + \sum_i \psi_i l_i + \sum_i \psi_i \varepsilon_{it} <$$

$$< \left(\sum_i \psi_i |v_{i1}|\right) d_{1t} + ... + \left(\sum_i \psi_i |v_{iJ}|\right) d_{Jt} + \sum_i \psi_i l_i + \sum_i \psi_i \varepsilon_{it} =$$

$$= \left(\sum_i \psi_i |v_{i1}| + ... + \sum_i \psi_i |v_{iJ}|\right) \left\{ \frac{\left(\sum_i \psi_i |v_{i1}|\right) d_{1t} + ... + \left(\sum_i \psi_i |v_{iJ}|\right) d_{Jt}}{\sum_i \psi_i |v_{i1}| + ... + \sum_i \psi_i |v_{iJ}|} \right\} + \sum_i \psi_i l_i +$$

$$\sum_i \psi_i \varepsilon_{it} =$$

$$= r_p D_{aggr} + \sum_i \psi_i l_i + \sum_i \psi_i \varepsilon_{it}$$

Note that here aggregating over the elements of the price vector we obtain the demand function in terms of the price index.This is an example of economic interpretation of the aggregation procedure that we propose in our paper, in particular, to assigning weights to the endogenous variables.

Derivation of the supply curve for the price index.

First, we write the aggregate (over all supplier types) supply equation:

$$\sum_h s_t^h = \sum_h g^h + \sum_h n^h \hat{E}_{t-1}^h p_t = \sum_h g^h + \left(\sum_h n^h\right) \hat{E}_{t-1}^{aggreg} p_t$$

Then we write the expression for each component of the vector: aggregate supply of each product equations. So, for each product j,

$$\sum_h s_{jt}^h = \sum_h g_j^h + \left(\sum_h n^h \hat{E}_{t-1}^h p_t\right)_j = \sum_h g_j^h + \left(n_{11}^1 + ... + n_{JJ}^S\right) \hat{E}_{t-1}^{aggreg} p_t^j.$$

Next, we aggregate over all supply equations using weights ψ_j.Aggregating across endogenous variables (prices) to get price index, we finally get the supply curve for the aggregated model

$$\hat{E}_{t-1}^{aggreg} P_t. = \left(\frac{\psi_1}{n_{11}^1 + ... n_{11}^S}\right) \sum_h s_{1t}^h + ... + \left(\frac{\psi_J}{n_{JJ}^1 + ... n_{JJ}^S}\right) \sum_h s_{Jt}^h -$$

$$-\sum_j \psi_j \left(\sum_h g_j^h / \left(n_{11}^1 + ... + n_{JJ}^S\right)\right) =$$

[7] To get this function, we aggregate the individual demand functions, not the reduced form equations (in which case we would obtain the equation for the intertemporal equilibrium price index).

$$= \left(\frac{\psi_1}{n_{11}^1 + ... n_{11}^S} + ... + \frac{\psi_J}{n_{JJ}^1 + ... n_{JJ}^S}\right) \left\{ \frac{\left(\frac{\psi_1}{n_{11}^1 + ... n_{11}^S}\right) \sum_h s_{1t}^h + ... + \left(\frac{\psi_J}{n_{JJ}^1 + ... n_{JJ}^S}\right) \sum_h s_{Jt}^h}{\left(\frac{\psi_1}{n_{11}^1 + ... n_{11}^S}\right) + ... + \left(\frac{\psi_J}{n_{JJ}^1 + ... n_{JJ}^S}\right)} \right\} -$$

$$- \sum_j \psi_j \left(\sum_h g_j^h / \left(n_{11}^1 + ... + n_{JJ}^S\right) \right) =$$

$$= r_m S_{aggr} - \sum_j \psi_j \left(\sum_h g_j^h / \left(n_{11}^1 + ... + n_{JJ}^S\right) \right).$$

It is clear that $\frac{\psi_j}{n_{jj}^1 + ... n_{jj}^S} > \frac{\psi_j}{S \max_h \left\{n_j^h\right\}}$.So, condition of aggregated economy stability in this class of models, $\sum_i \psi_i |v_{ij}| < \frac{\psi_j}{S \max_h \left\{n_{jj}^h\right\}}, \forall j$, is condition for E-stability of the aggregated cobweb model ($r_m > r_p$).

6 Conclusion

Based on the analysis of negative diagonal dominance, the alternative definition of D–stability, and the characteristic equation, we have been able to derive two groups of sufficient conditions and one group of necessary conditions for δ–stability.

Our paper partly resolves the open question posed by Honkapohja and Mitra (2005). As has been mentioned, Honkapohja and Mitra (2005) provide a general stability condition (criterion) for the case of persistently heterogeneous learning — a joint restriction on matrices of structural parameters and degrees of inertia, which implies that stability in such an economy is determined by the interaction of structural heterogeneity and learning heterogeneity. For the general (multivariate) case, however, it was not possible to derive stability conditions expressed in terms of an economy aggregated only across agent types.

They have derived sufficient conditions in terms of the structure of the economy, but this condition is very general: D–stability and H–stability of the structural matrices. We go a bit further in this direction by providing sufficient conditions for D–stability in terms of the aggregate economy structure.

In particular, using negative diagonal dominance (sufficient for D–stability) and our concept of aggregating an economy (both across agent types and endogenous variables), we have obtained sufficient conditions for δ–stability expressed in terms of E–stability of the aggregated economy and its structure. These were summarized as the “aggregated economy” sufficient conditions. One of them, in principle, can serve as a rule of thumb for checking a model for delta-stability.

We have found a unifying condition for the most general case of heterogeneous learning in linear forward–looking models. It is quite restrictive, of course. But our main achivement was to have shown that such a simple condition with E–stability meaning of some aggregated economy (notion, that already proved to be useful as a condition of stability under heterogeneous learning in previous learning literature) does exist for a large class of models with any finite natural numbers of agents, exogenous and endogenous variables.

The economic example provided in the end of the paper demonstrates the application of the aggregate economy conditions.

Next, based on the analysis of the alternative definition of D-stability, we have obtained sufficient conditions on the structure of the economy summarized as the "equal sign conditions".

Further, based on the analysis of the characteristic equation and requirement for negativity of all eigenvalues (necessary and sufficient for stability), we have derived a group of necessary conditions. Their failure can be used as an indicator of non–δ–stability.

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A Appendix

Here we provide the reader with definitions and theorems adapted from mathematics literature that are used for deriving conditions of $\delta-$stability. These results are structured according to the approach which is used for deriving stability conditions.

General definition of stability and D-stability of a matrix

Definition 13 *Matrix A is stable if all the solutions of the system of ordinary differential equations $\dot{x}(t) = Ax(t)$ converge toward zero as t converges to infinity.*

Theorem 14 *Matrix A is stable if and only if all its eigenvalues have negative real parts*

Definition 15 *($D-$stability) Matrix A is $D-$stable if DA is stable for any positive diagonal matrix D. A COMMENT FROM JOHNSON(1974), p.54: "Thus the D-stables are just those matrices which remain stable under any relative reweighting of the rows or columns."*

Lyapunov theorem approach

Theorem 16 *(Lyapunov): A real $n \times n$ matrix A is a stable matrix if and only if there exists a positive definite matrix H such that $A'H + HA$ is negative definite.*

Theorem 17 *(Arrow-McManus, 1958): Matrix A is D-stable if there exists a positive diagonal matrix C such that $A'C + CA$ is negative definite.*

Negative diagonal dominance approach

Definition 18 *(introduced by McKenzie): A real $n \times n$ matrix A is dominant diagonal if there exist n real numbers $d_j > 0, j = 1, ..., n,$ such that $d_j|a_{jj}| > \sum d_i|a_{ij}| : i \neq j), \ j = 1, \ldots, n$ This is called "column" diagonal dominance. "Row" diagonal dominance is defined as the existence of $d_i > 0$ such that $d_i|a_{ii}| > \sum d_j|a_{ij}| : j \neq i), i = 1, \ldots, n.$*

Theorem 19 *(sufficient condition for stability, McKenzie, 1960): If an $n \times n$ matrix A is dominant diagonal and the diagonal is composed of negative elements ($a_{ii} < 0$, all $i = 1, \ldots, n$), then the real parts of all its eigenvalues are negative, i.e. A is stable.*

Corollary 20 *If A has negative diagonal dominance, then it is D-stable.*

Characteristic equation approach

Theorem 21 *(Routh-Hurwitz necessary and sufficient conditions for negativity of eigenvalues of a matrix)*

Consider the characteristic equation

$|\lambda I - A| = \lambda^n + b_1\lambda^{n-1} + ... + b_{n-1}\lambda + b_n = 0$

determining the n eigenvalues λ of a real $n \times n$ matrix A, where I is the identity matrix. Then the eigenvalues λ all have negative real parts if and only if

$\Delta_1 > 0, \Delta_2 > 0, ..., \Delta_n > 0,$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \cdots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & b_{2k-6} & \cdots & b_k \end{vmatrix}.$$

Alternative definition of $D-$stability approach

Theorem 22 *(From Observation (iv) in Johnson (1974)). Consider $M_n(C)$, the set of all complex $n \times n$ matrices, D_n, the set of all $n \times n$ diagonal matrices with positive diagonal entries. Take $A \in M_n(C)$ and suppose that there is an $F \in D_n$ such that FA is stable. Then A is D–stable if and only if $A \pm iD$ is non–singular for all $D \in D_n$. If $A \in M_n(R)$, – the set of all $n \times n$ real matrices, then "$\pm$" in the above condition may be replaced with "+" since, for a real matrix, any complex eigenvalues come in conjugate pairs.*

B Appendix

Proof of Proposition 2

Use "columns" negative diagonal dominance of Ω which is sufficient for negative real parts of eigenvalues of $D\Omega$, look for condition which is sufficient for negative diagonal dominance in our structure[8]. Use as weight $diag\left(\varphi_1(\psi_1, ..., \psi_n), ..., \varphi_s\left(\psi_1, ..., \psi_n\right)\right)$,

[8] Note here again that if we assume that matrices F and D_w are diagonal, then we do not have to find a separate condition for stability of $D_w\Omega_F$. So this condition is sufficient for stability of the whole system.

$\varphi_i > 0,\, \psi_h > 0,\, \sum_i \psi_i = 1, \sum_h \varphi_h = 1.$

Take any block h and any column j.

$a^h_{jj} - 1 < 0$ -negative diagonal;

$\varphi_h \psi_j \left| a^h_{jj} - 1 \right| > (\varphi_1 + ... + \varphi_s) \sum_i \psi_i \left| a^h_{ij} \right| - \varphi_h \psi_j \left| a^h_{jj} \right| \; \forall j, h$

$\Updownarrow$

$a^h_{jj} - 1 < 0$

$-\varphi_h \psi_j a^h_{jj} + \varphi_h \psi_j > (\varphi_1 + ... + \varphi_s) \sum_i \psi_i \left| a^h_{ij} \right| - \varphi_h \psi_j \left| a^h_{jj} \right| \; \forall j, h$

$\Updownarrow$

$(Case 1)$ *and* $(Case 2)$

Case 1 $1 > a^h_{jj} \geqslant 0$

$\sum_i \psi_i \left| a^h_{ij} \right| < \frac{\varphi_h \psi_j}{\varphi_1 + ... + \varphi_S} \; \forall j, h$ such that $1 > a^h_{jj} > 0$

Case 2.$a^h_{jj} < 0$

$\sum_i \psi_i \left| a^h_{ij} \right| < \frac{\varphi_h \psi_j}{\varphi_1 + ... + \varphi_S} - \frac{2 \varphi_h \psi_j}{\varphi_1 + ... + \varphi_S} a^h_{jj} \; \forall j, h$ such that $a^h_{jj} < 0$

Using that in second case we have $a^h_{jj} < 0$, one may formulate the following sufficient condition$\sum_i \psi_i \left| a^h_{ij} \right| < \frac{\varphi_h \psi_j}{\varphi_1 + ... + \varphi_S} = \varphi_h \psi_j \; \forall j, h$ (The condition $1 > a^h_{jj}$ is implied by this relation, and the condition of case 2 is also satisfied). This is the condition of proposition 1. $Q.E.D.$

Proof of Proposition 3.

For proof of maximal beta–coefficient $\beta^1_{\max}$. take $\psi_i = \frac{1}{n}$.The previous general sufficient condition is rewritten as $\sum_i \left| a^h_{ij} \right| < \phi_h \; \forall j, h$. Now one is left to notice that condition $\beta^4_{\max} = \sum_h \max_j \left\{ \sum_i \left| a^h_{ij} \right| \right\} < 1$ implies that there exists such weights $\phi = (\phi_1, ..., \phi_S) > 0$ that above condition holds true which is sufficient for negative diagonal dominance of $\Omega.Q.E.D.$

For proof of maximal beta–coefficient $\beta^4_{\max}$. take $\varphi_h = \frac{1}{S}$.The previous general sufficient condition is rewritten as $\sum_i \psi_i \left| a^h_{ij} \right| < \frac{\psi_j}{S} \; \forall j, h$. Notice that condition $\beta^1_{\max} = S \sum_j \max_{h,i} \left| a^h_{ij} \right| < 1$ implies that there exist such weights $\psi = (\psi_1, .., \psi_n) > 0$ that above condition holds true which is sufficient for negative diagonal dominance of Ω. $Q.E.D.$

For proof of maximal beta–coefficients $\beta^2_{\max}$ and $\beta^3_{\max}$, use first "rows" negative diagonal dominance of Ω which is sufficient for negative real parts of eigenvalues of $D\Omega$, look for condition which is sufficient for negative diagonal dominance in our structure. Use as weight $diag\left(d_1, ..., d_n, ..., d_1, ..., d_n\right)$, $d_i > 0$, $\sum_i d_i = 1$

Take any block h and any row i.

$a_{ii}^h - 1 < 0$ -negative diagonal;

$d_i \left|a_{ii}^h - 1\right| > \sum_h \sum_j d_j \left|a_{ij}^h\right| - d_i \left|a_{ii}^h\right| \; \forall i, h$

$\Updownarrow$

$a_{ii}^h - 1 < 0$

$-d_i a_{ii}^h + d_i > \sum_h \sum_j d_j \left|a_{ij}^h\right| - d_i \left|a_{ii}^h\right| \; \forall i, h$

$\Updownarrow$

$(Case1)\ and\ (Case2)$

Case 1 $1 > a_{ii}^h \geqslant 0$

$\sum_h \sum_j d_j \left|a_{ij}^h\right| < d_i \; \forall i, h$ such that $1 > a_{ii}^h > 0$

Case 2.$a_{ii}^h < 0$

$\sum_h \sum_j d_j \left|a_{ij}^h\right| < d_i - 2 d_i a_{ii}^h \; \forall i, h$ such that $a_{ii}^h < 0$

Using that in second case we have $a_{ii}^h < 0$, one may formulate the following sufficient condition$\sum_h \sum_j d_j \left|a_{ij}^h\right| < d_i \; \forall i$ (The condition $1 > a_{ii}^h$ is implied by this relation, and the condition of case 2 is also satisfied). Notice that condition $\beta_{\max}^3 = S \sum_i \max_{h,j} \left|a_{ij}^h\right| < 1$ implies that there exist such weights $d = (d_1, .., d_n, ..., d_1, ..., d_n) > 0$ that above condition holds true which is sufficient for negative diagonal dominance of Ω. For proof of maximal beta–coefficient $\beta_{\max}^2$, take $d_i = \frac{1}{n}$ in above.The previous general sufficient condition is rewritten as $\sum_h \sum_j \left|a_{ij}^h\right| < 1 \; \forall i$. Now one is left to notice that condition $\beta_{\max}^2 = \max_i \sum_h \sum_j \left|a_{ij}^h\right| < 1$ implies that above condition holds true which is sufficient for negative diagonal dominance of $\Omega . Q.E.D.$

C Appendix

Proof of Proposition 6:

For case $n = 1$, the condition for alternative definition of D-stability simplifies to

$\Omega-$stable and

at least one of the following holds true

$\left(\frac{\frac{1}{\delta_1}}{1+\frac{1}{\delta_1^2}}(-A_1)+...+\frac{\frac{1}{\delta_S}}{1+\frac{1}{\delta_S^2}}(-A_S)\right)\neq 0,$

$\left(\frac{1}{1+\frac{1}{\delta_1^2}}(-A_1)+...+\frac{1}{1+\frac{1}{\delta_S^2}}(-A_S)+1\right)\neq 0.$

The first "equal sign" condition in Proposition 6 follows directly from the first inequality. The second condition that follows from the second inequality is proved below.

Necessity: Fix some subeconomy $(i_1,...,i_p)$ and force deltas corresponding to other agents to zero, while deltas for agents in the subeconomy to infinity. The resulting inequality will look like $-A_{i_1}-A_{i_2}-...-A_{i_p}+1\geq 0$ or $-A_{i_1}-A_{i_2}-...-A_{i_p}+1\leq 0$ for any subeconomy $(i_1,...,i_p)$. Note that each single economy has to satisfy $-A_i+1\geq 0$, and at least one of them has to have strict inequality, $-A_{i_0}+1>0$, - the result that follows from the requirement on the trace of the matrix $D(-\Omega)$ to be greater than zero (see proof of proposition 9 and 10). Thus we proved the necessity part of the second condition in proposition 6.

Sufficiency: We have $-A_{i_1}-A_{i_2}-...-A_{i_p}+1\geq 0$ for any subeconomy $(i_1,...,i_p)$ and $\exists j_0\vdots -A_{j_0}+1>0$, and have to prove that $\left(\frac{1}{1+\frac{1}{\delta_1^2}}(-A_1)+...+\frac{1}{1+\frac{1}{\delta_S^2}}(-A_S)+1\right)\neq 0$.

We group separately the terms corresponding to non-positive A_i's and terms corresponding to non-negative A_i's (arbitraly put zero A_i's to any of the groups).

Schematically, we will have

$$\underbrace{\left[\frac{1}{1+\frac{1}{\delta_1^2}}\left(A_1^-\right)+...+\frac{1}{1+\frac{1}{\delta_k^2}}\left(A_k^-\right)\right]}_{\leq 0}+\underbrace{\left[\frac{1}{1+\frac{1}{\delta_1^2}}\left(A_1^+\right)+...+\frac{1}{1+\frac{1}{\delta_m^2}}\left(A_m^+\right)\right]}_{\leq 1}-1.$$

If j_0 is such that $A_{j_0}<0$, then the first term is strictly negative. If j_0 is such that $1>A_{j_0}>0$, then the second term is strictly less than 1. In any case the whole expression is negative. (We used that $0<\frac{1}{1+\frac{1}{\delta_1^2}}<1$). So, the sufficiency part of the second condition in proposition 6 is proved.

Proof of Propositions 7 and 8:

For case $n=2$, the inequality in alternative definition of D–stability looks the following way:

$\det\left[\frac{-A_1}{1+\frac{i}{\delta_1}}+...+\frac{-A_S}{1+\frac{i}{\delta_S}}+I\right]=1+\det\frac{(-A_1)}{1+\frac{i}{\delta_1}}+...+\det\frac{(-A_S)}{1+\frac{i}{\delta_S}}+\frac{M_1(-A_1)}{1+\frac{i}{\delta_1}}+...+$

$$+\frac{M_1(-A_S)}{1+\frac{i}{\delta_S}}+\det mix\left(\frac{-A_1}{1+\frac{i}{\delta_1}},\frac{-A_2}{1+\frac{i}{\delta_2}}\right)+...+\det mix\left(\frac{-A_{S-1}}{1+\frac{i}{\delta_{S-1}}},\frac{-A_S}{1+\frac{i}{\delta_S}}\right)=$$

$$=1+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)^2\det(-A_1)+...+\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)^2\det(-A_S)+$$

$$+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)M_1(-A_1)+...+\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)M_1(-A_S)+...+$$

$$+\left(\frac{1-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}\right)\left(\frac{1-\frac{i}{\delta_2}}{1+\frac{1}{\delta_2^2}}\right)[\det mix(-A_1,-A_2)+\det mix(-A_2,-A_1)]+...+$$

$$+\left(\frac{1-\frac{i}{\delta_{S-1}}}{1+\frac{1}{\delta_{S-1}^2}}\right)\left(\frac{1-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}\right)[\det mix(-A_{S-1},-A_S)+\det mix(-A_S,-A_{S-1})]\neq 0$$

Taking real and imagianary part, one gets

$$\operatorname{Re}\det\left[\frac{-A_1}{1+\frac{i}{\delta_1}}+...+\frac{-A_S}{1+\frac{i}{\delta_S}}+I\right]=1+\frac{1-\frac{1}{\delta_1^2}}{\left(1+\frac{1}{\delta_1^2}\right)^2}\det(-A_1)+...+$$

$$+\frac{1-\frac{1}{\delta_S^2}}{\left(1+\frac{1}{\delta_S^2}\right)^2}\det(-A_S)+\frac{1}{1+\frac{1}{\delta_1^2}}M_1(-A_1)+...+\frac{1}{1+\frac{1}{\delta_S^2}}M_1(-A_S)+...+$$

$$+\frac{1-\frac{1}{\delta_1\delta_2}}{\left(1+\frac{1}{\delta_1^2}\right)\left(1+\frac{1}{\delta_2^2}\right)}[\det mix(-A_1,-A_2)+\det mix(-A_2,-A_1)]+...+$$

$$+\frac{1-\frac{1}{\delta_{S-1}\delta_S}}{\left(1+\frac{1}{\delta_{S-1}^2}\right)\left(1+\frac{1}{\delta_S^2}\right)}[\det mix(-A_{S-1},-A_S)+\det mix(-A_S,-A_{S-1})]$$

$$\operatorname{Im}\det\left[\frac{-A_1}{1+\frac{i}{\delta_1}}+...+\frac{-A_S}{1+\frac{i}{\delta_S}}+I\right]=\frac{-\frac{2i}{\delta_1}}{\left(1+\frac{1}{\delta_1^2}\right)^2}\det(-A_1)+...+$$

$$+\frac{-\frac{2i}{\delta_S}}{\left(1+\frac{1}{\delta_S^2}\right)^2}\det(-A_S)+\frac{-\frac{i}{\delta_1}}{1+\frac{1}{\delta_1^2}}M_1(-A_1)+...+\frac{-\frac{i}{\delta_S}}{1+\frac{1}{\delta_S^2}}M_1(-A_S)+...+$$

$$+\frac{-i\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)}{\left(1+\frac{1}{\delta_1^2}\right)\left(1+\frac{1}{\delta_2^2}\right)}[\det mix(-A_1,-A_2)+\det mix(-A_2,-A_1)]+...+$$

$$+\frac{-i\left(\frac{1}{\delta_{S-1}}+\frac{1}{\delta_S}\right)}{\left(1+\frac{1}{\delta_{S-1}^2}\right)\left(1+\frac{1}{\delta_S^2}\right)}[\det mix(-A_{S-1},-A_S)+\det mix(-A_S,-A_{S-1})]$$

From Im part of the determinant we see "equal sign" sufficient condition for this case:

$\det(-A_i)>0, [\det mix(-A_i,-A_j)+\det mix(-A_j,-A_i)]>0, i\neq j, M_1(-A_i)>0$ and $\Omega-$stable

or

$\det(-A_i)<0, [\det mix(-A_i,-A_j)+\det mix(-A_j,-A_i)]<0, i\neq j, M_1(-A_i)<0$ and $\Omega-$stable

Proof of Proposition 9 and 10:

We consider $\Gamma=D(-\Omega)$. Necessary and sufficient condition for stability of this matrix is that real parts of eigenvalues of $D(-\Omega)$ be greater than zero. And for the condition on eigenvalues to hold true it is necessary that all sums of principal minors of $D\left(-\Omega\right)$ grouped by the same size be greater than zero.

Really, characterisitic equation for eigenvalues of Γ has the form

$\det\left(\Gamma + I\mu\right) = \det\Gamma + \mu M_{n-1} + \mu^2 M_{n-2} + ... + \mu^{n-1} M_1 + \mu^n = 0$, where $\lambda = -\mu$ is the eigenvalue of Γ. and M_k is the sum of all principal minors of Γ of size k.

On the other hand, the same characteristic equation can be written in terms of the product decomposition of the polynomial:

$$(\mu + \lambda_1)\cdots(\mu + \lambda_n) = \underbrace{\lambda_1 ... \lambda_n}_{>0} + ... + \mu^{n-2} \underbrace{(\lambda_1\lambda_2 + ... + \lambda_{n-1}\lambda_n)}_{>0} + \mu^{n-1} \underbrace{(\lambda_1 + ... + \lambda_n)}_{>0} + \mu^n = 0.$$

Thus, all $M_k > 0$.

By writing down this condition in terms of $D(-\Omega)$, one gets that in each size group the sum of minors is subdivided into groups of sums of minors that contain the same number of columns of each block of $(-\Omega)$, i.e. $A_i - I$. The coefficient before such particular sum has the form $\left(\delta_{i_1}\right)^{j_1}\left(\delta_{i_2}\right)^{j_2} ... \left(\delta_{i_p}\right)^{j_p}$. This coefficient uniquely specifies the sum of minors by the size, the number of columns from each block, and from which subeconomy it is formed, $i_1, ..., i_p$. The size of the minors in such a group is equal to the total power of the coefficients, $j_1 + ... + j_p$, and the subscripts of deltas mean from which block of $(-\Omega)$ columns are taken, while the power of each delta indicates how many columns are taken from this particular block.

Let us fix one subeconomy (let us say, formed by blocks 1, 2, 3) and consider the limit of the inequalities for sum of minors, with deltas for other blocks going to zero. Doing the same operation for all subeconomies, we will get condition (*). The statement in proposition 10 is derived by setting all deltas for all subeconomies in condition (*) equal to 1.