The term structure of inflation risk premia and macroeconomic dynamics*

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Second draft: November 2005

Abstract

This paper estimates the size and dynamics of inflation risk premia in the euro area using information from nominal and index-linked yields. Our main result is that the inflation risk premium on long-term nominal yields is nonnegligible from an economic viewpoint. Break-even inflation rates – i.e. the difference between nominal and real yields with identical maturity – therefore represent a relatively crude approximation of inflation expectations. Break-even inflation rates are also found to contain useful information to forecast inflation and output growth, even when taking into account standard indicators such as the slope of the yield curve.

JEL classification: E43, E44

Keywords: Term structure of interest rates, risk premia, policy rules.

*The opinions expressed are personal and should not be attributed to the European Central Bank.

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1 Introduction

What determines the level of long-term nominal yields? Ideally, such level could be decomposed in a real yield, long term inflation expectations and an inflation risk premium.[1] Based on such a decomposition, central banks often interpret the difference between nominal and inflation-linked yields as a noisy measure of expected inflation, often called the "break-even inflation rate." The measure is noisy because it includes an inflation risk premium component. The main objective of this paper is to estimate the size of the inflation risk premium in euro area yields and to analyze its macroeconomic determinants.

The interest of central banks in break-even inflation rates is due to the fact that these rates can be viewed as a measure of credibility of the central bank's inflation objective. The reasoning goes as follows. If the objective is well-known because of a public announcement, as in the case of the European Central Bank, and if it is credible, it should be reflected in inflation expectations over horizons far into the future. In other words, any current inflationary shocks should be temporary and long-run inflation expectations should remain anchored at the level consistent with the announced objective.

This story, however, does not account for inflation risk premia. If such premia were time varying, they could explain observed variability of break-even inflation rates even if expected inflation always remained constant. A similar "credibility" story could, however, be told in terms of inflation risk premia. If the central bank's objective is credible, uncertainty on long-run inflationary developments should be relatively contained and inflation risk premia should be small. The link between credibility and inflation risk premia presumes that changes in such premia are predominantly due to uncertainty on the perceived inflation objective. For risk premia, this is not necessarily the case. Other sources of inflationary shocks, if very volatile or persistent, could have an impact on inflation risk premia at long maturities. Depending on their determinants, variations in the break-even inflation rate could reflect the underlying uncertainty of the economic environment, rather than being associated with lack of credibility of the inflation objective.

In order to disentangle these determinants formally, a necessary condition is a joint model of macroeconomic and term structure dynamics. Only within a macroeconomic model can notions such as "inflation target" be defined. Only if bonds are build on a

[1] We are here disregarding a convexity term, which is however likely to be small from a quantitative viewpoint.

macroeconomic framework can one discuss the impact on yields of inflationary shocks of various sources. Finally, a macro model should also provide a more realistic description of inflation dynamics, compared to a reduced-form model.

For these reasons, we adopt the framework developed in Hördahl, Tristani and Vestin (2005a), which in turns builds on Ang and Piazzesi (2003). More specifically, we price yields based on the dynamics of the short rate obtained from the solution of a linear macro model and using an essentially affine stochastic discount factor (see Duffie and Kan, 1996; Dai and Singleton, 2000; Duffee, 2002).[2]

Compared to the alternative of relying on a rich, microfounded model in the DSGE tradition, our modelling strategy has the advantage of being able to generate time-varying risk-premia while remaining highly tractable. With respect to smaller models which can be solved nonlinearly, our approach has the advantage of being independent of special assumptions imposed for analytical tractability, and of relying on a well-established monetary policy transmission mechanism. The drawback is obviously that we are unable to draw a link from the prices of risk to individuals' preferences.

Consistently with the essentially affine term structure literature, our specification of prices of risk is relatively flexible. If we want to be able to discriminate between the inflation risk premium and other risk premium components in nominal yields, the information provided by index-linked bonds is especially useful. However, index-linked yields are only available in the euro area since around 1999, which is too short a sample to permit econometric estimation and inference. In our empirical application, we therefore use a longer sample and proceed as follows: up to September 1999, we only use data for nominal yields and macroeconomic variables for the model estimation, while index-linked bonds as treated as unobservables; as of October 1999, the dataset is extended to include index-linked yields. Since parameter values do not change over time, the information incorporated in real yields in the second part of the sample also helps to discipline our estimates for the first part. Assuming absence of structural breaks – admittedly a strong assumption in the euro area case – this should produce superior estimates of inflation risk premia.

Our estimated real yields for the period before index-linked bonds were available also allows us to test whether the model-implied break-even inflation rate, raw or corrected

[2] Other recent papers that jointly model macroeconomic and nominal term structure dynamics include Dewachter and Lyrio (2004) and Rudebusch and Wu (2004).

for inflation risk premia, can provide useful information in forecasting macroeconomic variables. More specifically, one would expect break-even inflation rates to be positively correlated with actual future inflation, since they represent a noisy estimate of inflation expectations. On the other hand, if such expectations were consistent with those of the central bank, they may tend to trigger a policy response that, in turn, could have repercussions on future output growth.

Focusing on the 10-year maturity, our main result is that the inflation risk premium on nominal yields is positive and nonnegligible from an economic viewpoint. Break-even inflation rates represent therefore a relatively crude approximation of inflation expectations over this horizon. However, the most important determinant of fluctuations in this premium appear to be shocks to the perceived inflation objective. Hence, the interpretation of fluctuations in long-term break-even inflation rates in terms of policy credibility is not unwarranted.

Break-even inflation also helps to forecast future output growth over and above the contribution of other variables such as the policy rate, the nominal slope, and lagged output growth. Estimates from predictive growth regressions are consistent with the notion that increases in the 10-year break-even rate are associated with expected future inflation, and hence with future policy tightenings which slow down the economy. Such restrictive policy response, however, appears to be moderate, in the sense that it does not prevent a temporary increase in future inflation. Moreover, we find that break-even inflation rates are also useful predictors of future inflation, even when accounting for other standard variables.

Our paper is organized as follows. The next section discusses in more detail the advantage of our methodology to estimate the euro area inflation risk premium, both from a theoretical viewpoint and in relation to evidence available for other countries. Our zero-coupon real rates derived from index-linked bonds yields are presented in Section 3, where we also present some descriptive statistics on our full dataset of real and nominal bonds and macroeconomic variables. Section 4 outlines our model, its implications for the inflation risk premium and our econometric methodology. Our empirical results are presented in Section 5, where we show our estimates of the inflation risk premium at various maturities and also decompose it in terms of its macroeconomic determinants. In Section 6 we present our results on forecasting output and inflation using raw and risk-adjusted break-even inflation rates. Section 7 concludes.

2 What should we expect on inflation risk premia?

It goes without saying that we are not first to analyze the inflation risk premium in nominal bonds. However, there is little agreement in the theoretical and empirical literature on the size and even the sign of the premium. The raw evidence available from index-linked bonds points to a positive difference between nominal and real yields, and the nominal yield curve also appears to be steeper than the real yield curve (e.g. Roll, 2004). In order to make inference on the inflation risk premium, however, one needs to take a stance on inflation expectations over the life of the bond. Since the latter are also unobservable, a theoretical framework is necessary to answer the question in the title of this section.

From a theoretical viewpoint, it is clear at least from Fischer (1975) that there is no reason to expect the inflation risk premium to be positive. The sign of the premium depends entirely on the covariance between real returns on nominal bonds and the stochastic discount factor. In simple microfounded models, the log stochastic discount factor is proportional to consumption growth and the inflation risk premium will be positive when consumption growth and inflation are negatively correlated. In more general set-ups, however, this simple intuition is lost. In the approximate solution of a calibrated model with habit persistence and nominal rigidities, Hördahl, Tristani and Vestin (2005b) argue that the average inflation risk premium in the US should be positive, but small.

A number of recent empirical studies suggest that the inflation risk premium in the U.S. nominal term structure should be positive and non-negligible in economic terms. Buraschi and Jiltsov (2005) use a monetary version of a real business cycle model to characterize and estimate the inflation risk premium, and find an average premium of 15 basis points at the 1-month horizon and 70 basis points at the ten-year horizon. Based on an essentially affine term structure model with regime switching, Ang and Bekaert (2005) also find positive inflation risk premia of a comparable magnitude in the US, ranging from zero for short term bonds to almost 100 basis points for 5-year bonds. These papers, however, do not incorporate information from inflation-indexed bonds. Based on an essentially affine set-up which incorporates index-linked UK yields, Risa (2001) also finds a positive inflation risk premium, but on average this is downward sloping in maturity: it is equal to 2.2% for a theoretical instantaneous bond and it falls to 1.7% for a 20-year bond. The short term inflation risk premium is also much more volatile than the long term premium. An even starker difference characterizes the results in Evans (2003), where the UK term structure is modelled using a regime switching set-up which incorporates information from index-linked bonds. Evans (2003) also finds a downward sloping inflation risk premium, but this is large and negative for most maturities, reaching -1.8% or even -3.5% at the 10-year horizon depending on the prevailing state.

All in all, there appear to be no robust results on the sign, size, maturity structure and volatility of inflation risk premia. The different results in the literature could partly be due to differences in samples or country.

3 Data

Our main objective is to extract long-term inflation expectations and premia from the term structure of euro area interest rates. In order to achieve this goal, however, we face a number of limitations.

First and foremost, we need to deal with the possibility that the creation of the single European currency, the euro, induced a structural break in economic relationships. If the structural break did occur, we would obviously need to disregard pre-1999 data, which would make our model estimates extremely imprecise given the short sample period left. Other work on European data, however, has not identified significant changes in estimated macroeconomic relationships (e.g. Smets and Wouters, 2003). For this reason, we proceed "as if" the structural break did not occur, in either structural macro and policy parameters, or in investors' attitude towards risk.

A second difficulty that we face, compared to macro studies, is related to the inclusion of yields in our dataset. The aforementioned studies rely on "synthetic" data for the pre-1999 period, namely data obtained by aggregation of national series based on fixed weights. Given our emphasis on absence-of-arbitrage restrictions on yield dynamics, we cannot follows this route, since average bond prices would violate arbitrage by construction.

For these reasons, we use German data for the pre-EMU period. We therefore assume that investors' perceptions of the monetary policy rule followed by the Bundesbank before, and the ECB after 1999 could be characterized by the same functional form and the same parameters. We also assume that the dynamics of the German aggregate output and inflation can be described by the same laws of motion as the corresponding macroeconomic variables for the euro area. The only form of change that we allow from pre-EMU to EMU years is in the perceived inflation objective, which is allowed to vary over time. These are obviously strong assumptions. To increase their plausibility, we start our estimations in 1991, i.e. we focus on a period which, except the ERM crisis, was characterized by a strong convergence of nominal interest rates at all maturities in euro area countries.

3.1 Index-linked zeros

For our analysis, we first derive zero-coupon equivalent rates from index-linked prices and coupons. Specifically, we rely on data for index-linked bonds issued by the French Treasury. In this process, as is typically the case in the literature, we abstract from tax and liquidity issues. Concerning liquidity, in particular, there is no clear evidence of a positive liquidity premium on European index-linked bonds. In spite of the paucity of available issues during the initial months of the index-linked market, these bonds were in fact met immediately by strong investors' demand.

We also assume that index-linked bonds are truly risk-free, i.e. we dismiss the inflation risk borne by investors because of the indexation lags. In principle, we could use the methodology of Kandel, Ofer and Sarig (1996) and Evans (1998) to account for such lag. However, Evans (1998) estimates the indexation-lag premium to be quite small, notably around 1.5 basis points, in the UK, where the indexation lag is of 8 months. Since the lag is of only 3 months in the euro area, we believe that any estimate of the indexation-lag premium would be well within the range of any measurement error.

Finally, we face the constraint that only bonds indexed to the French CPI, rather than the euro area HICP, were available up to late 2001, when the French Treasury began issuing index-linked bonds linked to the euro area HICP. While the difference between euro area HICP and French CPI is not huge, it is persistent over time, so that yields on HICP-linked bonds consistently tended to be below those of CPI-linked bonds. Since the variable of interest for monetary policy, and hence affecting the short term rate, is the HICP, we use a mixed series in our estimation: the HICP-linked bond as of October 2002 and the CPI-based series prior to this.[3] However, we adjust CPI-linked zeros downwards

[3] While HICP-linked bonds were introduced already in 2001, sufficient data to allow estimation of zero-

by an amount equal to the average difference between CPI and HICP-linked yields at each maturity.

In order to construct zero-coupon equivalents for index-linked yields, we follow the spline method in McCulloch and Kochin (2000). The methodology is designed to work with yield data that are only available for few maturities. It is based on a discount function of the form

$$\delta(m) = \exp\left[-\sum_{j=1}^{n} a_j \psi_j(m)\right],$$

where m is the time to maturity and n is the number of maturities available from the data, while the $\psi_j(m)$'s are splines defined by

$$\psi_j(m) = \theta_j(m) - \frac{\theta_j''(m_n)}{\theta_{n+1}''(m_n)} \theta_{n+1}(m), \quad j = 1, ..., n,$$

and the functions $\theta_j(m)$ are given by

$$\begin{aligned}
\theta_1(m) &= m \\
\theta_2(m) &= m^2 \\
\theta_j(m) &= \max(0, m - m_{j-2})^3, \quad j = 3, ..., n+1.
\end{aligned}$$

The resulting zero-coupon yields for the 3-year, 5-year and 10-year maturities are shown in Figure 1. The real zeros are relatively high in 2000 and 2001, when growth was also relatively high, and lower in more recent years. More precisely, at the 10-year maturity real yields vary between 4.0 and 1.5 percent.

3.2 Nominal yields and macro data

We can use these real rates to construct break-even inflation rates for the corresponding maturities, namely the straightforward difference between nominal and real yields. For nominal yields, we use zero-coupon yields derived from German government bond data.

Figure 2 shows break-even inflation rates for 3, 5 and 10-year maturities. Since 1999, break-even rates have varied within a close range. At the 10-year maturity, in particular, they have mostly oscillated between 1 and 2 percent.

coupon real yields is available only as of October 2002.

Based on the intuition that break-even inflation rates reflect future inflation expectations and risk premia, one might expect fluctuations in these rates to be correlated with actual inflation. We therefore analyze the correlations of break-even rates with inflation and output. Inflation and output data are also related to Germany up until 1998 and to the euro area afterwards. More specifically, inflation is defined as the monthly log-change in German CPI until 1998, in the HICP for the euro area afterwards. For output, we use log-industrial production – German first, then related to the euro area. Following Clarida, Galí and Gertler (1998), our output gap series is defined in terms of deviations of industrial production from a quadratic trend.

The correlations with inflation and output growth or the output gap of break-even inflation rates, the nominal slope and the levels of nominal and real rates are reported in Table 1. Break-even inflation rates are indeed positively correlated with inflation, but very little: the correlation is also insignificant from a statistical viewpoint. Break-even inflation rates are more strongly and significantly correlated with output growth: this may reflect the fact that stronger growth can induce fears of future inflationary pressures. This channel of transmission is explicitly accounted for in the model of section 4.

The correlation between break-even inflation rates and output growth is also present at various output leads and is therefore suggestive of some forecasting ability. The correlation is actually larger than for the yield spread, which is a well-known good predictor of future output developments (more recently Ang, Piazzesi and Wei, 2005, or, for the euro area, Moneta, 2003). A significant correlation with output growth is also present in nominal yields, while real yields are essentially uncorrelated with both output and inflation.

In order to specify our model of section 4, we also analyze whether long-term real rates appear to include information which is significantly different from that contained in nominal rates. For this purpose, we look at the principal components of nominal yields, of nominal and real yields, and of all yields plus our macro-variables. We can obviously carry out this analysis only for the period over which real yields are available, namely October 1999 to December 2004.

Over this sample, 3 principal components are necessary to capture 99% of the variance of nominal yields. As soon as we add the real yields, however, 4 principal components are needed. When we include macro-variables, 4 principal components continue to capture 99% of the variance of all variables, but the fourth becomes much more important: it explains 4% of the variance of the variables, compared to 1% explained in the case without macroeconomic variables.

4 Model

We rely on a structural economic model, which is specified directly at the aggregate level. This formulation has the advantage of avoiding to rely on a specific microfoundation, typically based on the existence of a representative household. The drawback is obviously that the model is a purely descriptive tool.

While more flexible than most microfounded versions, the model remains very stylized. It includes just two equations which describe the evolution of inflation, π_t, and the output gap, x_t. Since we are going to estimate the model at the monthly level, the two equations are specified with a relatively elaborate lead and lag structure:

$$\widehat{\pi}_t = \mu_\pi \frac{1}{12} \sum_{i=1}^{12} E_t \left[\widehat{\pi}_{t+i} \right] + (1 - \mu_\pi) \sum_{i=1}^{3} \delta_{\pi i} \widehat{\pi}_{t-i} + \delta_x \widehat{x}_t + \varepsilon_t^\pi \tag{1}$$

$$\widehat{x}_t = \frac{1}{12} \mu_x \sum_{i=1}^{12} E_t \left[\widehat{x}_{t+i} \right] + (1 - \mu_x) \sum_{i=1}^{3} \zeta_{xi} \widehat{x}_{t-i} - \zeta_r \left(\widehat{r}_t - E_t \left[\widehat{\pi}_{t+1} \right] \right) + \varepsilon_t^x \tag{2}$$

where π_t is inflation, defined as the monthly change in the log-price level, x_t is the output gap, r_t the 1-month nominal interest rate, and the hats denote deviations from the mean. The specification of the model is similar to that in Hördahl, Tristani and Vestin (2005a), and it is motivated by the literature on the so-called "new-Keynesian" Phillips curve (e.g. Gali' and Gertler, 1999) and on estimation of consumption-Euler equations (e.g. Fuhrer, 2000). Both equations include a forward-looking term capturing expectations over the next year of inflation and output, respectively. The 3 lags in the backward-looking components of the two equations are motivated empirically. In the estimation, we impose $\mu_\pi + (1 - \mu_\pi) \sum_i \delta_{\pi i} = 1$, a version of the natural rate hypothesis.

The simple representation of the economy in equations (1) and (2) incorporates explicitly some standard channels of transmission of inflationary shocks and of monetary policy. Inflation can be due to demand shocks ε_t^x, which increase output above potential and create excess demand, and to cost-push shocks ε_t^π, which increase prices without demand pressures. In turn, monetary policy can affect inflation via stimuli or restrictions of aggregate demand, i.e. modifying the real interest rate $\widehat{r}_t - E_t \left[\widehat{\pi}_{t+1} \right]$, or influencing inflation expectations.

To solve for the rational expectations equilibrium, we need an assumption on how monetary policy is conducted. We focus on private agents' perceptions of the monetary policy rule followed by the central banks, which is supposedly to set the nominal short rate according to

$$\widehat{r}_t = (1-\rho)\left(\beta\left(\frac{1}{12}E_t\left[\sum_{i=0}^{11}\widehat{\pi}_{t+i}\right] - \widehat{\pi}_t^*\right) + \gamma\widehat{x}_t\right) + \rho\widehat{r}_{t-1} + \eta_t \tag{3}$$

where $\widehat{\pi}_t^*$ is the perceived inflation target and η_t is a "monetary policy shock".

This is consistent with the formulation in Clarida, Galí and Gertler (2000). The first two terms represent a forward-looking Taylor-type rule, where the rate responds to deviations of expected inflation from the inflation target. The second part of the rule is motivated by interest rate smoothing concerns, i.e. the desire to avoid producing large volatility in nominal interest rates.

We also allow for a time-varying, rather than constant, inflation target π_t^*. We adopt this formulation in order to allow for some evolution in the behavior of monetary policy over time, in particular in the changeover from the Bundesbank to the ECB. While this cannot account for all possible sorts of structural breaks in the policy rule after 1999, it has the advantage of not requiring a full re-estimation of the model over the short euro area period.

Finally, we need to specify the processes followed by the stochastic variables of the model, i.e. the perceived inflation target and the three structural shocks. We assume that our 3 macro shocks are serially uncorrelated and normally distributed with constant variance. The only factor that we allow to be serially correlated is the unobservable inflation target, which will follow an AR(1) process

$$\widehat{\pi}_t^* = \phi_\pi \widehat{\pi}_{t-1}^* + u_{\pi,t} \tag{4}$$

where $u_{\pi,t}$ is a normal disturbance with constant variance uncorrelated with the other structural shocks.

4.1 Building the term structure

In order to solve the model we write it in the general form

$$\begin{bmatrix} \mathbf{X}_{1,t+1} \\ E_t\mathbf{X}_{2,t+1} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix} + \mathbf{K}\widehat{r}_t + \begin{bmatrix} \Sigma\xi_{1,t+1} \\ \mathbf{0} \end{bmatrix}, \tag{5}$$

where $\mathbf{X}_1$ is the vector of predetermined variables, $\mathbf{X}_2$ includes the variables which are not predetermined, $\widehat{r}_t$ is the policy instrument and ξ_1 is a vector of independent, normally distributed shocks. The short-term rate can be written in the feedback form

$$\widehat{r}_t = -\mathbf{F} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix}. \tag{6}$$

The solution of the (5)-(6) model can be obtained numerically following standard methods. We choose the methodology described in Söderlind (1999), which is based on the Schur decomposition. The result are two matrices $\mathbf{M}$ and $\mathbf{C}$ such that $\mathbf{X}_{1,t} = \mathbf{M}\mathbf{X}_{1,t-1} + \Sigma\xi_{1,t}$ and $\mathbf{X}_{2,t} = \mathbf{C}\mathbf{X}_{1,t}$.[4] Consequently, the equilibrium short term interest rate will be equal to $\widehat{r}_t = \mathbf{\Delta}'\mathbf{X}_{1,t}$, where $\mathbf{\Delta}' \equiv -(\mathbf{F}_1+\mathbf{F}_2\mathbf{C})$ and $\mathbf{F}_1$ and $\mathbf{F}_2$ are partitions of $\mathbf{F}$ conformable with $\mathbf{X}_{1,t}$ and $\mathbf{X}_{2,t}$. Focusing on the short-term (policy) interest rate, the solution can be written as

$$\begin{aligned} \widehat{r}_t &= \mathbf{\Delta}'\mathbf{X}_{1,t} \\ \mathbf{X}_{1,t} &= \mathbf{M}\mathbf{X}_{1,t-1} + \Sigma\xi_{1,t}. \end{aligned} \tag{7}$$

The system (7) expresses the short term interest rate as a linear function of the vector $\mathbf{X}_1$, which in turn follows a first order Gaussian VAR. Both the short rate equation and the law of motion of vector $\mathbf{X}_1$ have been obtained endogenously, as functions of the parameters of the macroeconomic model. This contrasts with the standard affine set-up based on unobservable variables, where both the short rate equation and the law of motion of the state variables are postulated exogenously.

To derive the term structure, we only need to impose the assumption of absence of arbitrage opportunities, which guarantees the existence of a risk neutral measure, and to specify a process for the stochastic discount factor. Following the essentially affine

[4] The presence of non-predetermined variables in the model implies that there may be multiple solutions for some parameter values. We constrain the system to be determinate in the iterative process of maximizing the likelihood function.

formulation (see Duffee, 2002; Dai and Singleton, 2002), an important element of the stochastic discount factor will be the market prices of risk λ_t, which will be affine in the vector $\mathbf{X}_{1t}$, i.e. $\lambda_t = \widetilde{\lambda}_0 + \widetilde{\lambda}_1 \mathbf{X}_{1t}$. Note that $\mathbf{X}_{1t}$ includes the 4 stochastic factors of the system, i.e. the inflation target and the three white noise shocks. These shocks will induce risk premia, but in the essentially affine formulation the premia will also depend on the level of the other states. Since our $\mathbf{X}_{1t}$ includes 11 variables – the four stochastic factors plus 3 lags of the output gap and inflation and 1 lag of the short term rate – the maximum number of non-zero elements in the $\widetilde{\lambda}_1$ matrix is 4×11.

Estimation of 44 parameters just for the state-dependent prices of risk is prohibitive. We therefore impose some restrictions on the λ_1 matrix. More specifically, rather than allowing the market prices of risk to be independently influenced by the lags of the macroeconomic variables, we impose that such lag-dependence is induced by the current levels of those macro variables. For example, we assume that the lags of inflation will potentially affect the the prices of risk only through their effect on current inflation, output, or the nominal interest rate. This assumption implies that we can rewrite the market prices of risk as linear functions of only $\widehat{x}_t$, $\widehat{r}_t$, $\widehat{\pi}_t$ and $\widehat{\pi}^*_t$. Since each of these variables can be written as a a linear combination of the vector of predetermined variables using the model's solution, this assumption is equivalent to imposing cross-restrictions on the elements of the $\widetilde{\lambda}_1$ matrix.

More precisely, we first define a new vector $\mathbf{Z}_t \equiv [\widehat{\pi}^*_t, \widehat{x}_t, \widehat{\pi}_t, \widehat{r}_t]'$ and then rewrite the solution equation for the short term interest rate as a function of $\mathbf{Z}_t$, $r_t = \overline{\mathbf{\Delta}}' \mathbf{Z}_t$. The $\mathbf{Z}_t$ vector can obviously be expressed as a linear combination of the predetermined variables using the solution $\mathbf{X}_{2,t} = \mathbf{C}\mathbf{X}_{1,t}$, so that $\mathbf{Z}_t = \hat{\mathbf{D}}\mathbf{X}_{1,t}$ for a suitably defined matrix $\hat{\mathbf{D}}$. The (nominal) pricing kernel m_{t+1} is defined as $m_{t+1} = \exp(-r_t)\, \psi_{t+1}/\psi_t$, where ψ_{t+1} is the Radon-Nikodym derivative assumed to follow the log-normal process $\psi_{t+1} = \psi_t \exp\left(-\frac{1}{2}\lambda'_t \lambda_t - \lambda'_t \xi_{1,t+1}\right)$. Finally, market prices of risk are assumed to be affine in the state vector $\mathbf{Z}_t$

$$\lambda_t = \lambda_0 + \lambda_1 \mathbf{Z}_t, \tag{8}$$

so that the size of λ_1 will be simply 4×4. Since $\mathbf{Z}_t = \hat{\mathbf{D}}\mathbf{X}_{1,t}$, $\lambda_1 \mathbf{Z}_t = \lambda_1 \hat{\mathbf{D}}\mathbf{X}_{1,t}$ and λ_1 will induce restrictions on $\widetilde{\lambda}_1$ such that $\widetilde{\lambda}_1 \hat{\mathbf{D}}^{-1} = \lambda_1$. In the estimation, we restrict λ_1 further to include only the 8 elements in its first two columns.[5]

[5] To ensure conformability with the other matrices which affect the prices of bonds, we actually define

In the appendix we show that the reduced form (7) of our macroeconomic model, coupled with the aforementioned assumptions on the pricing kernel, implies that the continuously compounded yield y_t^n on a zero coupon nominal bond with maturity n is given by

$$y_t^n = A_n + B_n' \mathbf{Z}_t, \tag{9}$$

where the A_n and B_n' matrices can be derived using recursive relations. Stacking all yields in a vector $\mathbf{Y}_t$, we write the above equations jointly as $\mathbf{Y}_t = \mathbf{A} + \mathbf{B}'\mathbf{Z}_t$ or, equivalently, $\mathbf{Y}_t = \mathbf{A}_n + \tilde{\mathbf{B}}_n' \mathbf{X}_{1,t}$, where $\tilde{\mathbf{B}}_n' \equiv \mathbf{B}_n' \hat{\mathbf{D}}$. Similarly, for real bonds y_t^{*n} we obtain

$$y_t^{*n} = A_n^* + B_n'^* \mathbf{Z}_t, \tag{10}$$

4.2 The inflation risk premium in our model

In this model, there is no simple real-monetary dichotomy: all monetary shocks have real consequences. Care must therefore be taken to define inflation risk premia, as opposed to premia associated to real types of risk.

In this sense, it is instructive to first look at the inflation risk premium which characterizes the short term rate. Given the nominal and real short rates, r_t and r_t^* respectively, the appendix shows that the former can be written as

$$r_t = r_t^* + E_t\left[\pi_{t+1}\right] + prem_{\pi,t} + \tfrac{1}{2}\mathbf{C}_\pi \Sigma \Sigma' \mathbf{C}_\pi' \tag{11}$$

where

$$\begin{aligned} r_t^* &= \mathbf{C}_\pi \Sigma \left(\lambda_0 - \tfrac{1}{2}\Sigma' \mathbf{C}_\pi'\right) + \left(\overline{\boldsymbol{\Delta}}' - \mathbf{C}_\pi \left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t \\ E_t\left[\pi_{t+1}\right] &= \mathbf{C}_\pi \mathbf{M}\hat{\mathbf{D}}^{-1}\mathbf{Z}_t \\ prem_{\pi,t}^1 &= -\mathbf{C}_\pi \Sigma \lambda_0 - \mathbf{C}_\pi \Sigma \lambda_1 \mathbf{Z}_t \end{aligned}$$

We define $prem_{\pi,t}$ as the inflation risk premium to distinguish it from the convexity term $\frac{1}{2}\mathbf{C}_\pi \Sigma\Sigma' \mathbf{C}_\pi'$, which would affect the short term rate even if the prices of risk were zero.

$\mathbf{Z}_t$ to be of the same size of $\mathbf{X}_{1t}$. As a result, $\hat{\mathbf{D}}$ and λ_1 are $n_1 \times n_1$ matrix, but their additional elements are all zero.

The inflation risk premium is related to the full standard deviation of inflation, the term $\mathbf{C}_\pi\Sigma$, irrespective of the actual shock that determines it. For given prices of risk, the inflation risk premium will be higher, the higher the variance of the shocks, and the higher their impact on inflation.

For bonds of other maturities, a more complex expression holds (see the appendix). More specifically, the state-dependent part of the inflation risk premium in nominal yields can be written as

$$prem_{\pi,t}^{ytm,n} = const. + \mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \frac{\sum_{i=1}^{n} \widehat{\mathbf{M}}^i}{n} \mathbf{Z}_t \qquad (12)$$

where $\widehat{\mathbf{M}} \equiv \hat{\mathbf{D}} \left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1 \right)$ captures the risk-adjustment in the law of motion of the transformed state vector $\mathbf{Z}_t$.

Depending on the prices of risk, the matrix $\widehat{\mathbf{M}}$ could have eigenvalues outside the unit circle even if $\mathbf{M}$ does not. If its eigenvalues are within the unit circle, inflation risk premia on long term yields will be bounded from above. Long term premia will also be more sensitive to changes in the states $\mathbf{Z}_t$ than premia on short term bond, because $\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n$ tends to increase as n increases. If, instead, the risk-adjusted law of motion is non-stationary, i.e. if some of the eigenvalues of $\widehat{\mathbf{M}}$ are outside the unit circle, then the sum in equation (12) is not bounded and inflation risk premia can play an even larger role on long term yields.

The time-varying component of the inflation risk premium can also be written in deviation from the model-expected average inflation over the maturity of the bond, i.e. $\overline{\pi}_{t+n} = \frac{1}{n}\sum_{i=1}^{n} \pi_{t+i}$. It follows that

$$prem_{\pi,t}^{ytm,n} = const. + E_t\overline{\pi}_{t+n} + \mathbf{C}_\pi \left(\hat{\mathbf{D}}^{-1} \frac{\sum_{i=1}^{n} \widehat{\mathbf{M}}^i}{n} - \frac{\sum_{i=1}^{n} \mathbf{M}^i}{n} \hat{\mathbf{D}}^{-1} \right) \mathbf{Z}_t$$

which emphasizes that the inflation risk premium arises because of the difference between the historical and risk-adjusted laws of motions of the state vector $\mathbf{Z}_t$.

4.3 Maximum likelihood estimation

In order to estimate the model, we need to distinguish first between observable and unobservable variables in the $\mathbf{X}_{1t}$ vector. We adopt the approach which is common in the finance literature and which involves inverting the relationship between yields and unobservable factors (Chen and Scott, 1993). We also use the common approach of assuming that some of the yields are imperfectly measured to prevent stochastic singularity. More precisely, we use yields on 1, 3, 6-month, 1, 3, 7-year nominal bonds and on 3, 5, 7, 10 real bonds. We assume that all bonds are imperfectly observable, with the exception of nominal bonds at the 3-month and 3-year maturities.

To deal with the lack of data on real yields up until 1999, we simply treat such yields as unobservable variables. Since these are not state variables, their unobservability has no impact on the likelihood. They are included in the measurement equation as of October 1999 through their impact on the measurement errors. The likelihood function can therefore be written as

$$\begin{aligned}\pounds(\theta) = &-(T-1)\left(\ln|J| + \frac{n_p}{2}\ln(2\pi) + \frac{1}{2}\ln\left|\mathbf{\Sigma\Sigma}'\right| + \frac{n_m}{2}\ln(2\pi) + \frac{1}{2}\sum_{i=1}^{n_m}\ln\sigma_{m,i}^2\right) \\ &-\frac{1}{2}\sum_{t=2}^{T}\left(\mathbf{X}_{1,t}^u - \mathbf{M}^u\mathbf{X}_{1,t-1}^u\right)'\left(\mathbf{\Sigma\Sigma}'\right)^{-1}\left(\mathbf{X}_{1,t}^u - \mathbf{M}^u\mathbf{X}_{1,t-1}^u\right) - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{n_m}\frac{\left(u_{t,i}^m\right)^2}{\sigma_{m,i}^2} \\ &-(T-t_r)\left(\frac{n_r}{2}\ln(2\pi) + \frac{1}{2}\sum_{i=1}^{n_r}\ln\sigma_{r,i}^2\right) - \frac{1}{2}\sum_{t=t_r}^{T}\sum_{i=1}^{n_r}\frac{\left(u_{t,i}^r\right)^2}{\sigma_{r,i}^2}\end{aligned}$$

where $\mathbf{X}_{1,t}^u$ are the unobservable variables included in the $\mathbf{X}_{1,t}$ vector, u_t^m are the measurement error shocks, $\mathbf{J}$ is a Jacobian matrix defined in the appendix, $\mathbf{\Sigma\Sigma}'$ is the variance-covariance matrix of the four macroeconomic shocks, σ_i are the standard deviations of measurement error shocks, T is the sample size, t_r is the observation from which index-linked yields are available, n_m and n_r are the numbers of measurement errors in nominal and real bonds, respectively, and n_p is the number of variables measured without error.

The problem of maximizing the likelihood is nontrivial, given the large size of the parameter space. We employ the method of simulated annealing, introduced to the econometric literature by Goffe, Ferrier and Rogers (1994). The method is developed with an aim towards applications where there may be a large number of local optima. One disadvantage of the simulated annealing method is that it does not provide us with an estimate of the derivatives, evaluated at the maximum, of the likelihood function with respect to the parameter vector, i.e. $\partial\ln(\pounds(\theta))/\partial\theta'$. These derivatives are necessary to compute asymptotic estimates of the variance-covariance matrix of the parameters. The derivatives could be evaluated numerically, but their computation is sensitive to the selection of an arbitrary step-lenght $\partial\theta$. To deal with this problem, we follow Anderson *et al.* (1996) and rely on analytical results to calculate the Jacobian $\partial \ln (\pounds(\theta)) / \partial \theta'$.

5 The term structure of inflation risk premia in the euro area

5.1 Parameter estimates and impulse responses

An advantage of our approach is that the parameters which affect the historical dynamics of the state vector can be interpreted economically. Table 2 reports our preferred estimates over the 1991-2004 period. We select this period because, with the exception of the months around the ERM crisis, it was characterized by an increased convergence of nominal interest rates at all maturities in euro area countries, and it should therefore be more suited for our assumption that monetary policy and interest rates dynamics could be characterized by the same equations as for the EMU period. This period also has the advantage of excluding the German reunification episode.

We estimate a policy rule which is largely consistent with standard macroeconomic estimates. The rule is characterized by a high degree of interest rate smoothing, a mild response to inflation deviations from the objective, and an output gap response which is essentially zero. The degree of forward lookingness of the output gap equation is relatively small, while it is high for the inflation equation. The latter result is different from many other estimates (see e.g. Jondeau and Le Bihan, 2001), including our own for the 1975-1998 period (see Hördahl, Tristani and Vestin, 2005a). Rather than to the different sample period, it is mostly due to the different definition of inflation as monthly, rather than year-on-year, log-price changes. Intuitively, monthly inflation is much less persistent than year-on-year inflation, thus the lesser role of backward-looking elements. The standard deviation of the fundamental shocks is relatively low, suggesting that the model is capable of accounting endogenously for a large part of macroeconomic dynamics.

The standard deviation of measurement errors is also broadly consistent with the results of affine models without macroeconomic variables. Errors are slightly larger for real yields, which occasionally – notably at the very end of the sample period – translates into significant mispricings.

Figures 4a-d show impulse response functions of the macroeconomic variables and of the break-even inflation rate to the four macroeconomic shocks of the system. More specifically, a policy shock, which amounts to an increase by 25 basis points of the short term rate, generates a decline in the output gap and a protracted reduction in inflation. As a result, inflation expectations fall and so does the break-even inflation rate at all maturities, but by an amount which is decreasing in maturity.

A positive shock to the inflation objective has strong inflationary consequences. Output increases persistently and inflation overshoots the new objective, so that the short-term nominal interest rate also increases. Break-even inflation rates also increase, the more so for shorter maturities.

An inflation shock has initially a mildly expansionary effect, because it induces an initial fall in real rates. After approximately 6 months, however, the restrictive policy stance generates a mild recession, which eventually brings inflation back to the baseline. Break-even inflation rates tend to increase temporarily, but the movement is almost negligible and very short-lived at the 10-year horizon.

Finally, a positive output gap has protracted inflationary consequences, which also bring about a prolonged increase in policy interest rates. As a result, short-term real rates tend to rise, but break-even inflation rates are essentially unchanged at medium and long horizons.

Finally, Figure 5 illustrates our model's ability to capture the actual dynamics in the short-term rate. It presents a historical decomposition of the shocks that generated deviations in the nominal interest rate compared to the path which this was expected to follow in January 1991. An important determinant of the overall trend in the nominal interest rate appears to be a faster and deeper than expected reduction of the inflation objective from the higher levels of the beginning of the seventies. Inflation and especially output shocks also exert a significant influence on the nominal interest rate, consistently with the systematic behavior captured by the Taylor rule. Monetary policy shocks, i.e. deviations from the Taylor-rule benchmark, play a significant role especially at the beginning and at the end of the sample.

5.2 Macroeconomic determinants of the inflation risk premium

While the impulse responses of break-even inflation rates are qualitatively consistent with the projected evolution of inflation after all shocks, it is important to assess how large and how variable an inflation risk premium is. Our main conclusions are that the inflation premium at the 10-year horizon is nonnegligible from an economic viewpoint. Over the 2000-2004 period for which actual index-linked data are available, it is on average equal to 60 basis points at the 10-year horizon.

The time series of 3 and 10-year inflation risk premia are shown in Figure 6a. Since the introduction of the euro, the estimated 10-year inflation risk premium has fluctuated between around 20 and 100 basis points. It is estimated at between 90 and 100 basis points at the end of 2004. The 3-year inflation risk premium is estimated to have been slightly higher than the 10-year premium during this period.

In order to make sense of the evolution of the 10-year inflation risk premium, we decompose it into its determinants in Figure 6b. The premium is insensitive to the evolution of inflation and the output gap. Its time variation is mostly linked to changes in the perceived inflation objective, and to a lesser extent to changes in the short term rate. More specifically, the compensation required by investors against the risk of any inflationary shock is larger when the inflation objective is below its long run value (just below 2% in the model), since a low objective is associated to expectations of an eventual increase in inflation.

A specular way to view the inflation risk premium is via the calculation of risk-adjusted break-even inflation rates, which provide a correct, model-consistent measure of inflation expectations over the life of the bond. Figure 7 shows a time series of risk-adjusted 10-year break-even inflation rates together with the inflation target. The target is generally decreasing over the nineties, from somewhat higher values achieved immediately after German unification. It hovers between 2 and 1.5 percent during the EMU years.

The evolution of expected inflation over the next 10 years broadly matches that of the inflation objective. The two series, however, are not identical. More specifically, expected inflation can vary significantly from the current estimate of the inflation objective. This discrepancy reflects the persistent nature of the dynamics generated by shocks to the inflation objective. From the impulse responses, it is clear that inflation overshoots the objective when this increases. When the objective is low, as is estimated to be during 2004, inflation expectations are even lower, and are expected to remain low persistently over time. As a result, the model produces a surprising combination of relatively high inflation risk premium and relatively low expected inflation.

6 Forecasting output with the break-even inflation rate

Given that we can generate model-consistent estimates of break-even inflation rates for the entire sample - even during the period before index-liked bonds were issued - and that we can disentangle the inflation premium component from the expected inflation component, it is of interest to examine whether these variables are useful in forecasting future output and inflation. In order to investigate this, we run regressions of these and other variables on future output growth and inflation, for various maturities. The other variables we include are the 10-year - 1-month slope of the nominal term structure, which has been shown empirically to be useful in forecasting inflation and economic activity, lagged values of the respective dependent variable, as well as the short-term (one-month) nominal interest rate, which Ang et al. (2005) find has substantial predictive power for future output growth.

In different regressions, we check whether the raw, unadjusted 10-year break-even inflation rate, the premium-corrected break-even rate, and the inflation premium implied by our model each have additional predictive power for the macro variables we consider. Table 3 shows the results for output growth 12, 24 and 36 months ahead. While none of the variables are significant at the shortest horizon, it is clear from the table that all three model-implied variables help to forecast future output growth over and above the contribution of the other variables at longer horizons. The negative parameter estimates for the break-even rate in Table 3 are consistent with the notion that increases in the break-even rate are associated with expected future inflation. In turn, the latter may trigger a policy tightening which slows down future economic activity. This explanation is supported by the results in Panel (b) in Table 3, where risk-adjusted break-even rates are used.

Such restrictive policy response, however, does not appear to prevent a temporary increase in future inflation, as indicated in Panels (a) and (b) of Table 4. Specifically, an increase in the break-even rate is associated with an increase in future inflation, and this effect is statistically significant at horizons beyond one year. The same result applies for an increase in the premium-corrected break-even inflation rate. Interestingly, Panel (c) indicates that increases to the inflation premium are significantly associated with declining future inflation.

7 Conclusions

The difference between nominal and inflation-linked bond yields, the break-even inflation rate, is often used by central banks and others as an indicator of market expectations of future inflation. However, the break-even inflation rate is a noisy measure of expected inflation because it includes an inflation risk premium component. This paper uses information from both nominal and index-linked yields to estimate the size and dynamics of inflation risk premia in the euro area, in order to allow disentangling inflation expectations and inflation risk premia in break-even rates. This is done by adopting the macro-finance term structure framework developed in Hördahl, Tristani and Vestin (2005a), in which yields are based on the dynamics of the short rate obtained from the solution of a linear macro model, combined with an essentially affine stochastic discount factor. Apart from delivering estimates of the inflation risk premium, this approach has the advantage that it also makes it possible to analyze its macroeconomic determinants.

The main result is that the inflation risk premium on long-term nominal yields is nonnegligible from an economic viewpoint, and highly time-varying. Hence, break-even inflation rates represent a relatively crude approximation of inflation expectations, and changes in break-even rates cannot reliably be interpreted as changes in expected inflation. A decomposition of the inflation risk premium at the 10-year horizon shows that the time-variation is mostly linked to changes in the perceived inflation objective, and to a lesser extent to changes in the short-term interest rate.

Finally, break-even inflation rates are also found to contain useful information to forecast inflation and output growth, even when taking into account standard indicators such as the slope of the nominal term structure. Specifically, regressions of break-even rates on inflation or on output growth 24 and 36 months ahead result in negative parameter estimates. This suggests that increases in the break-even rate are associated with higher expected future inflation, which in turn may trigger a tightening of monetary policy and a subsequent slow-down in future economic activity.

A Appendix

A.1 Pricing real and nominal bonds

The solution of the macro-model is of the form

$$\mathbf{X}_{1,t+1} = \mathbf{M}\mathbf{X}_{1,t} + \Sigma\xi_{1,t+1},$$
$$\mathbf{X}_{2,t+1} = \mathbf{C}\mathbf{X}_{1,t+1},$$

where the nominal short term rate can be written as

$$\begin{aligned} r_t &= -\left(\mathbf{F}_1 + \mathbf{F}_2\mathbf{C}\right)\mathbf{X}_{1,t} \\ &\equiv \mathbf{\Delta}'\mathbf{X}_{1,t} \end{aligned}$$

with $\mathbf{F}_1$ and $\mathbf{F}_2$ partitions of $\mathbf{F}$ conformable with $\mathbf{X}_{1,t}$ and $\mathbf{X}_{2,t}$.

Alternatively, we can write this in terms of a transformed vector $\mathbf{Z}_t$ defined $\mathbf{Z}_t = \hat{\mathbf{D}}\mathbf{X}_{1,t}$, so that the short rate is

$$r_t = \overline{\mathbf{\Delta}}'\mathbf{Z}_t,$$

for a suitably defined matrix $\hat{\mathbf{D}}$. From the macro model solution, we also know that

$$\begin{aligned} \pi_{t+1} &= \mathbf{C}_\pi\mathbf{M}\mathbf{X}_{1,t} + \mathbf{C}_\pi\Sigma\xi_{1,t+1} \\ &= \mathbf{C}_\pi\mathbf{M}\hat{\mathbf{D}}^{-1}\mathbf{Z}_t + \mathbf{C}_\pi\Sigma\xi_{1,t+1} \end{aligned}$$

where $\mathbf{C}_\pi$ is the relevant row of $\mathbf{C}$.

Now assume that the real pricing kernel is m_{t+1}, so that the following fundamental asset pricing relation holds

$$E_t\left[m_{t+1}\left(1 + R_{t+1}\right)\right] = 1,$$

where R_{t+1} denotes the real return on some asset.

If we now want to price an n-period nominal bond, we get

$$\frac{p_t^n}{q_t} = E_t\left[m_{t+1}\frac{p_{t+1}^{n-1}}{q_{t+1}}\right],$$

where q_t is the price level in the economy. In terms of inflation rates, $\pi_{t+1} \equiv \ln q_{t+1} - \ln q_t$, this is

$$p_t^n = E_t\left[m_{t+1}\frac{p_{t+1}^{n-1}}{\exp\left(\pi_{t+1}\right)}\right].$$

Notice that this is equivalent to postulating a nominal pricing kernel $m_{t+1}^* \equiv m_{t+1}/\exp\left(\pi_{t+1}\right)$, such that

$$p_t^n = E_t\left[m_{t+1}^* p_{t+1}^{n-1}\right].$$

We thus define m_{t+1}^* as $m_{t+1}^* = \exp\left(-r_t\right)\frac{\psi_{t+1}}{\psi_t}$, where ψ_{t+1} is the Radon-Nikodym derivative ψ_{t+1} assumed to follow the lognormal process

$$\psi_{t+1} = \psi_t \exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\xi_{1,t+1}\right),$$

and where λ_t is the vector of market prices of risk associated with the underlying sources of uncertainty $\xi_{1,t+1}$ in the economy. We also assume that the market prices of risk are affine in the transformed state vector $\mathbf{Z}_t$,

$$\lambda_t = \lambda_0 + \lambda_1 \mathbf{Z}_t,$$

which leads to a standard solution for nominal bond prices as

$$Y_t = -\frac{A_n}{n} - \frac{B_n'}{n}\mathbf{Z}_t$$

where

$$\begin{aligned}
\bar{A}_{n+1} &= \bar{A}_n - \bar{B}_n'\hat{\mathbf{D}}\Sigma\lambda_0 + \tfrac{1}{2}\bar{B}_n'\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}_n \\
\bar{B}_{n+1}' &= \bar{B}_n'\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right) - \overline{\mathbf{\Delta}}'
\end{aligned}$$

A.1.1 Real bonds

From the definition of the pricing kernel implies

$$r_t = -\ln m_{t+1}^* - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\xi_{1,t+1}$$

which translates into a real pricing kernel

$$m_{t+1} = \exp\left(-r_t + \pi_{t+1}\right)\frac{\psi_{t+1}}{\psi_t}$$

or

$$m_{t+1} = \exp\left(-\mathbf{\Delta}'\mathbf{Z}_t + \mathbf{C}_\pi\mathbf{M}\mathbf{X}_{1,t} + \mathbf{C}_\pi\Sigma\xi_{1,t+1} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\xi_{1,t+1}\right)$$

We postulate again that real bond prices will be exponential-affine functions of the state variables,

$$p_t^n = \exp\left(\bar{A}_n^* + \bar{B}_n^{*\prime}\mathbf{Z}_t\right),$$

where $\bar{A}_n^*$ and $\bar{B}_n^*$ are parameters which depend on the maturity n. Taken together with the pricing kernel relation, this can be used to identify the structure of the bond pricing relation.

For an $n+1$ bond, we obtain

$$\begin{aligned}
p_t^{n+1} &= E_t\left[m_{t+1}p_{t+1}^n\right] \\
&= \exp\left(\bar{A}_n^* - \overline{\mathbf{\Delta}}'\mathbf{Z}_t + \mathbf{C}_\pi\mathbf{M}\mathbf{X}_{1,t} + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\mathbf{M}\mathbf{X}_{1,t} - \frac{1}{2}\lambda_t'\lambda_t\right) \\
&\quad E_t\left[\exp\left(\left(\mathbf{C}_\pi\Sigma - \lambda_t' + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\Sigma\right)\xi_{1,t+1}\right)\right]
\end{aligned}$$

where we used

$$\mathbf{Z}_{t+1} = \hat{\mathbf{D}}\mathbf{M}\mathbf{X}_{1,t} + \hat{\mathbf{D}}\Sigma\xi_{1,t+1}$$

Now note that

$$E_t\left[\exp\left(\left(\mathbf{C}_\pi\Sigma + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\Sigma - \lambda_t'\right)\xi_{1,t+1}\right)\right] = \exp\left(\tfrac{1}{2}\left(\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma - \lambda_t'\right)\left(\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma - \lambda_t'\right)'\right)$$

and rearrange terms to obtain

$$p_t^{n+1} = \exp\left(\begin{array}{c} \bar{A}_n^* + \frac{1}{2}\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma\Sigma'\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)' - \left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma\lambda_0 \\ + \left(\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right) - \overline{\mathbf{\Delta}}'\right)\mathbf{Z}_t \end{array}\right)$$

We can therefore identify $\bar{A}_n^*$ and $\bar{B}_n^*$ recursively, starting from the parameters obtained for nominal bonds, as

$$\begin{aligned} \bar{A}_{n+1}^* &= \bar{A}_n^* + \tfrac{1}{2}\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma\Sigma'\left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)' - \left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\Sigma\lambda_0 \\ \bar{B}_{n+1}^{*\prime} &= \left(\mathbf{C}_\pi + \bar{B}_n^{*\prime}\hat{\mathbf{D}}\right)\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right) - \overline{\mathbf{\Delta}}' \end{aligned}$$

For a 1-month real bond, in particular, we obtain

$$\begin{aligned} p_t^1 &= E_t\left[m_{t+1}\right] \\ &= \exp\left(\left(-\overline{\mathbf{\Delta}}' + \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t - \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma'\mathbf{C}_\pi'\right)\right) \end{aligned}$$

which can be used to start the recursion. Note that the short term real rate is

$$r_t^* = \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma'\mathbf{C}_\pi'\right) + \left(\overline{\mathbf{\Delta}}' - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t$$

A.2 Short-rates spread

The effect of the inflation risk premium is to drive a wedge between riskless real yields and ex-ante real yields, namely nominal yields net of expected inflation. For the short term rate, in particular, we can write

$$r_t = r_t^* + E_t\left[\pi_{t+1}\right] + prem_{\pi,t} + \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}_\pi'$$

where

$$\begin{aligned} r_t^* &= \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma'\mathbf{C}_\pi'\right) + \left(\overline{\mathbf{\Delta}}' - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t \\ E_t\left[\pi_{t+1}\right] &= \mathbf{C}_\pi\mathbf{M}\hat{\mathbf{D}}^{-1}\mathbf{Z}_t \\ prem_{\pi,t} &= -\mathbf{C}_\pi\Sigma\left(\lambda_0 + \lambda_1\mathbf{Z}_t\right) \end{aligned}$$

Note that the discrepancy between ex-ante real and risk-free rates is not only due to inflation risk, but also includes a convexity term $\frac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}_\pi'$. We define as inflation risk premium the component of the difference which would vanish if market prices of risk were zero.

A.3 Yields

For all maturities, recall that the continuously compounded yield is, for nominal and real bonds, respectively

$$
\begin{aligned}
y_{t,n} &= -\frac{\bar{A}_n}{n} - \frac{\bar{B}_n'}{n}\mathbf{Z}_t \\
y^*_{t,n} &= -\frac{\bar{A}^*_n}{n} - \frac{\bar{B}^{*\prime}_n}{n}\mathbf{Z}_t
\end{aligned}
$$

The yield spread is therefore simply

$$
y_{t,n} - y^*_{t,n} = -\frac{1}{n}\left(\bar{A}_n - \bar{A}^*_n\right) - \frac{1}{n}\left(\bar{B}_n' - \bar{B}^{*\prime}_n\right)\mathbf{Z}_t
$$

where

$$
\begin{aligned}
\bar{A}_{n+1} - \bar{A}^*_{n+1} &= \bar{A}_n - \bar{A}^*_n - \left(\bar{B}_n' - \bar{B}^{*\prime}_n\right)\hat{\mathbf{D}}\Sigma\lambda_0 + \mathbf{C}_\pi\Sigma\lambda_0 - \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}_\pi' \\
&\quad - \mathbf{C}_\pi\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_n + \tfrac{1}{2}\left(\bar{B}_n'\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}_n - \bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_n\right) \\
\bar{B}_{n+1}' - \bar{B}^{*\prime}_{n+1} &= \left(\bar{B}_n' - \bar{B}^{*\prime}_n\right)\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right) - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)
\end{aligned}
$$

Note that the nominal bond equation can be solved explicitly as

$$
\begin{aligned}
\bar{A}_n &= \bar{A}_1 + \sum_{i=1}^{n-1}\left(\tfrac{1}{2}\overline{\mathbf{B}}_i'\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\overline{\mathbf{B}}_i - \overline{\mathbf{B}}_i'\hat{\mathbf{D}}\Sigma\lambda_0\right), \\
\overline{B}_n' &= -\overline{\mathbf{\Delta}}'\sum_{i=0}^{n-1}\hat{\mathbf{D}}^i\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)^i
\end{aligned}
$$

Similar, for the real bond $\bar{A}^*_n$ we obtain

$$
\begin{aligned}
\bar{A}^*_n &= n\mathbf{C}_\pi\Sigma\left(\tfrac{1}{2}\Sigma'\mathbf{C}_\pi' - \lambda_0\right) + \sum_{i=1}^{n-1}\left(\bar{B}^{*\prime}_i\hat{\mathbf{D}}\Sigma\Sigma'\mathbf{C}_\pi' + \tfrac{1}{2}\bar{B}^{*\prime}_i\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_i - \bar{B}^{*\prime}_i\hat{\mathbf{D}}\Sigma\lambda_0\right) \\
\bar{B}^{*\prime}_n &= \left(\mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right) - \overline{\mathbf{\Delta}}'\right)\sum_{i=0}^{n-1}\hat{\mathbf{D}}^i\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)
\end{aligned}
$$

Note that the law of motion of the transformed state vector can be written as $\mathbf{Z}_{t+1} = \hat{\mathbf{D}}\mathbf{M}\hat{\mathbf{D}}^{-1}\mathbf{Z}_t + \hat{\mathbf{D}}\Sigma\xi_{1,t+1}$, so that the term $\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)$ represent the expected change in $\mathbf{Z}_t$ under Q. We can then define a new matrix $\widehat{\mathbf{M}} = \hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)$. Note also that the sum $\sum_{i=0}^{n-1}\widehat{\mathbf{M}}^i$ can be solved out as $\sum_{i=0}^{n-1}\widehat{\mathbf{M}}^i = \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n$ for bonds of finite maturity.[6]

[6] For bonds of infinite maturity, the sum will only be defined if all eigenvalues of $\widehat{\mathbf{M}}$ are inside the unit circle. This is not necessarily true, even if the eigenvalues of $\mathbf{M}$ are within the unit circle by construction.

It follows that and

$$\overline{B}_n' = -\overline{\mathbf{\Delta}}' \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n$$
$$\bar{B}_n^{*\prime} = \left(\mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} - \overline{\mathbf{\Delta}}'\right) \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n$$

and

$$\overline{B}_n' - \bar{B}_n^{*\prime} = -\mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n$$

Note also that

$$E_t \left[\pi_{t+n}\right] = \mathbf{C}_\pi \mathbf{M}^n \hat{\mathbf{D}}^{-1} \mathbf{Z}_t$$

and that expected average inflation up to $t+n$, $\overline{\pi}_{t+n}$ is

$$\begin{aligned} E_t \overline{\pi}_{t+n} &= \frac{1}{n} \sum_{i=1}^{n} E_t \pi_{t+i} \\ &= \mathbf{C}_\pi \frac{\sum_{i=1}^{n} \mathbf{M}^i}{n} \hat{\mathbf{D}}^{-1} \mathbf{Z}_t \end{aligned}$$

For $\bar{A}_n$ and $\bar{A}_n^*$ we obtain (for $n > 1$)

$$\begin{aligned} &\bar{A}_n - \bar{A}_n^* \\ &= -n \mathbf{C}_\pi \Sigma \left(\tfrac{1}{2} \Sigma' \mathbf{C}_\pi' - \lambda_0\right) \\ &- \tfrac{1}{2} \mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \left[\sum_{i=1}^{n-1} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^i \hat{\mathbf{D}} \Sigma \Sigma' \hat{\mathbf{D}}' \left(\mathbf{I} - \widehat{\mathbf{M}}'\right)^i\right] \left(\mathbf{I} - \widehat{\mathbf{M}}'\right)^{-1} \widehat{\mathbf{M}}' \left(\hat{\mathbf{D}}'\right)^{-1} \mathbf{C}_\pi' \\ &+ \mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \left[\sum_{i=1}^{n-1} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^i \hat{\mathbf{D}} \Sigma \Sigma' \hat{\mathbf{D}}' \left(\mathbf{I} - \widehat{\mathbf{M}}'\right)^i\right] \left(\mathbf{I} - \widehat{\mathbf{M}}'\right)^{-1} \overline{\mathbf{\Delta}} \\ &+ \mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \widehat{\mathbf{M}}^{-1} \left(\mathbf{I} - \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n\right) \left(\mathbf{I} - \widehat{\mathbf{M}}\right) \hat{\mathbf{D}} \Sigma \lambda_0 \\ &- \left(\mathbf{C}_\pi \hat{\mathbf{D}}^{-1} \widehat{\mathbf{M}} - \overline{\mathbf{\Delta}}'\right) \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1} \widehat{\mathbf{M}}^{-1} \left(\mathbf{I} - \left(\mathbf{I} - \widehat{\mathbf{M}}\right)^n\right) \left(\mathbf{I} - \widehat{\mathbf{M}}\right) \hat{\mathbf{D}} \Sigma \Sigma' \mathbf{C}_\pi' \end{aligned}$$

A.4 Holding period returns

We define the one-period holding period $e_{n,t}^*$ as

$$e_{n,t}^* = E_t \left[\ln p_{t+1,n-1} - \ln p_{t,n}\right]$$

Using the bonds equations, we know that

$$p_{t+1}^{n-1} = \exp\left(\bar{A}_{n-1}^* + \bar{B}_{n-1}^{*\prime} \mathbf{Z}_{t+1}\right)$$

and

$$\begin{aligned} e_{n,t}^* = &-\tfrac{1}{2} \left(\mathbf{C}_\pi + \bar{B}_{n-1}^{*\prime} \hat{\mathbf{D}}\right) \Sigma \Sigma' \left(\mathbf{C}_\pi + \bar{B}_{n-1}^{*\prime} \hat{\mathbf{D}}\right)' + \left(\mathbf{C}_\pi + \bar{B}_{n-1}^{*\prime} \hat{\mathbf{D}}\right) \Sigma \lambda_0 \\ &+ \left(\bar{B}_{n-1}^{*\prime} \hat{\mathbf{D}} \Sigma \lambda_1 - \mathbf{C}_\pi \left(\mathbf{M} \hat{\mathbf{D}}^{-1} - \Sigma \lambda_1\right) + \overline{\mathbf{\Delta}}'\right) \mathbf{Z}_t \end{aligned}$$

which in case of the 1-period bond collapses to

$$e^*_{1,t} = -\tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}'_\pi + \mathbf{C}_\pi\Sigma\lambda_0 + \left(\overline{\mathbf{\Delta}}' - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{-1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t$$

i.e. the short term rate.

The excess real holding period return is therefore

$$e^*_{n,t} - e^*_{1,t} = -\tfrac{1}{2}\bar{B}^{*\prime}_{n-1}\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_{n-1} + \bar{B}^{*\prime}_{n-1}\hat{\mathbf{D}}\Sigma\left(\lambda_0 - \Sigma'\mathbf{C}'_\pi\right) + \bar{B}^{*\prime}_{n-1}\hat{\mathbf{D}}\Sigma\lambda_1\mathbf{Z}_t$$

Similarly, for the nominal term structure we obtain

$$\begin{aligned} e_{n,t} &= -\tfrac{1}{2}\bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}_{n-1} + \bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\lambda_0 + \left(\bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\lambda_1 + \overline{\mathbf{\Delta}}'\right)\mathbf{Z}_t \\ e_{n,t} - e_{1,t} &= \bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma'\hat{\mathbf{D}}'\bar{B}_{n-1}\right) + \bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\lambda_1\mathbf{Z}_t \end{aligned}$$

so that the nominal-real spread net of expected inflation is

$$\begin{aligned} e_{n,t} - e^*_{n,t} - E_t\left[\pi_{t+1}\right] = &-\tfrac{1}{2}\left(\bar{B}'_{n-1}\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}_{n-1} - \bar{B}^{*\prime}_{n-1}\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_{n-1}\right) \\ &+ \mathbf{C}_\pi\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_{n-1} + \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}'_\pi \\ &+ \left(\left(\bar{B}'_{n-1} - \bar{B}^{*\prime}_{n-1}\right)\hat{\mathbf{D}} - \mathbf{C}_\pi\right)\left(\Sigma\lambda_0 + \Sigma\lambda_1\mathbf{Z}_t\right) \end{aligned}$$

Once again, we can rewrite this using the solutions for $\bar{B}'_{n-1}$ and $\bar{B}^{*\prime}_{n-1}$ to obtain

$$\begin{aligned} &e_{n,t} - e^*_{n,t} - E_t\left[\pi_{t+1}\right] \\ &= -\mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}}\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{M}}^{n-1}\right)\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\left(\mathbf{I} - \left(\widehat{\mathbf{M}}'\right)^{n-1}\right)\left(\mathbf{I} - \widehat{\mathbf{M}}'\right)^{-1}\left(\overline{\mathbf{\Delta}} - \tfrac{1}{2}\widehat{\mathbf{M}}'\left(\hat{\mathbf{D}}'\right)^{-1}\mathbf{C}'_\pi\right) \\ &+ \mathbf{C}_\pi\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_{n-1} + \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}'_\pi \\ &- \left(\mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}}\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{-1}\left(\mathbf{I} - \widehat{\mathbf{M}}^{n-1}\right)\hat{\mathbf{D}} + \mathbf{C}_\pi\right)\left(\Sigma\lambda_0 + \Sigma\lambda_1\mathbf{Z}_t\right) \end{aligned}$$

A.5 Forwards

Forwards are defined as

$$\begin{aligned} f^*_{n,t} &= \ln p^n_t - \ln p^{n+1}_t \\ &= \mathbf{C}_\pi\Sigma\lambda_0 - \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}'_\pi - \bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\left(\Sigma'\mathbf{C}'_\pi - \lambda_0\right) - \tfrac{1}{2}\bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_n \\ &+ \left(\bar{B}^{*\prime}_n\left(\mathbf{I} - \widehat{\mathbf{M}}\right) - \mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}} + \overline{\mathbf{\Delta}}'\right)\mathbf{Z}_t \end{aligned}$$

Note that

$$E_t r^*_{t+n} = \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma'\mathbf{C}'_\pi\right) + \left(\overline{\mathbf{\Delta}}' - \mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}}\right)\hat{\mathbf{D}}\mathbf{M}^{n-1}\hat{\mathbf{D}}^{-1}\mathbf{Z}_t$$

so that the real forward premium is

$$\begin{aligned} f^*_{n,t} - E_t r^*_{t+n} = &-\bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\left(\Sigma'\mathbf{C}'_\pi - \lambda_0\right) - \tfrac{1}{2}\bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_n \\ &+ \left(\bar{B}^{*\prime}_n - \bar{B}^{*\prime}_n\widehat{\mathbf{M}} + \left(\overline{\mathbf{\Delta}}' - \mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}}\right)\left(\mathbf{I} - \hat{\mathbf{D}}\mathbf{M}^{n-1}\hat{\mathbf{D}}^{-1}\right)\right)\mathbf{Z}_t \end{aligned}$$

The forward spread is

$$f_{n,t} - f^*_{n,t} = \left(\bar{B}'_n - \bar{B}^{*\prime}_n\right)\hat{\mathbf{D}}\Sigma\lambda_0 - \frac{1}{2}\bar{B}'_n\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}_n + \frac{1}{2}\bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\Sigma'\hat{\mathbf{D}}'\bar{B}^*_n - \mathbf{C}_\pi\Sigma\lambda_0 + \frac{1}{2}\mathbf{C}_\pi\Sigma\Sigma'\mathbf{C}'_\pi$$
$$+ \bar{B}^{*\prime}_n\hat{\mathbf{D}}\Sigma\Sigma'\mathbf{C}'_\pi + \left(\bar{B}'_n - \bar{B}^{*\prime}_n - \left(\bar{B}'_n - \bar{B}^{*\prime}_n\right)\widehat{\mathbf{M}} + \mathbf{C}_\pi\hat{\mathbf{D}}^{-1}\widehat{\mathbf{M}}\right)\mathbf{Z}_t$$

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Table 1: Correlations between yields and macro variables.

Correlations coeffs.	dy	x	π
$y_{t,36} - y^*_{t,36}$	**0.23**	**0.62**	0.14
$y_{t,60} - y^*_{t,60}$	**0.25**	**0.76**	0.14
$y_{t,120} - y^*_{t,120}$	**0.20**	**0.68**	0.05
$y_{t,36} - r_t$	**0.20**	**0.52**	0.00
$y_{t,60} - r_t$	0.16	**0.51**	−0.01
$y_{t,120} - r_t$	0.09	**0.45**	−0.01
$y_{t,36}$	**0.11**	−0.20	0.02
$y_{t,60}$	**0.12**	−0.19	0.02
$y_{t,120}$	**0.11**	−0.22	0.02
$y^*_{t,36}$	0.01	−.47	−0.04
$y^*_{t,60}$	0.00	−.48	−0.04
$y^*_{t,120}$	−0.02	−.44	−0.01

Sample: October 1999 to December 2004. Bold figures are significant at the 5 percent level (based on den Haan and Levin (2000) standard errors).

Table 2: Parameter estimates
(Sample period: Jan. 1991 - Dec. 2004)

Parameter	Point estimate	Standard error
ρ	0.946	0.016
β	1.356	0.612
γ	0.000	0.146
μ_π	0.740	0.058
δ_x	0.112	0.014
μ_x	0.289	0.069
ζ_r	0.047	0.040
ϕ_{π^*}	0.990	–
$\sigma_{\pi^*} \times 10^2$	0.056	0.004
$\sigma_\eta \times 10^2$	0.021	0.002
$\sigma_x \times 10^2$	0.277	0.016
$\sigma_\pi \times 10^2$	0.073	0.004
$\sigma_1^m \times 10^2$	0.022	0.003
$\sigma_2^m \times 10^2$	0.006	0.000
$\sigma_3^m \times 10^2$	0.011	0.001
$\sigma_4^m \times 10^2$	0.019	0.002
$\sigma_1^r \times 10^2$	0.013	0.004
$\sigma_2^r \times 10^2$	0.017	0.015
$\sigma_3^r \times 10^2$	0.019	0.020
$\sigma_4^r \times 10^2$	0.022	0.014
$\lambda_{0,1}$	−0.077	0.010
$\lambda_{0,2}$	−0.055	0.057
$\lambda_{0,3}$	0.266	0.398
$\lambda_{0,4}$	−0.761	0.121

$\lambda_1 \times 10^{-2}$

	π^*	r	π	x
π^*	−0.362 (0.057)	0.728 (0.087)	0	0
r	−0.260 (0.418)	−1.209 (0.972)	0	0
π	0.011 (0.395)	1.127 (0.478)	0	0
x	1.172 (0.520)	−5.352 (0.852)	0	0

Asymptotic standard errors are based on the outer-product estimate of the information matrix. The estimates of the lag coefficients for inflation and output are not reported.

Table 3: Parameter estimates for predictive output growth regressions

Horizon	(a) short rate	10y - 1m spread	lagged growth	10y. break-even
12 months	0.506 (1.558)	4.517 (3.863)	−0.014 (0.012)	−22.096 (17.749)
24 months	**1.409** (0.626)	**5.923** (1.109)	**−0.019** (0.005)	**−27.494** (4.293)
36 months	**0.998** (0.220)	**3.800** (0.191)	**−0.012** (0.001)	**−14.982** (1.518)

Horizon	(b) short rate	10y - 1m spread	lagged growth	corr. break-even
12 months	2.775 (3.237)	4.270 (3.591)	−0.014 (0.012)	−8.353 (6.547)
24 months	**4.038** (0.988)	**5.408** (1.061)	**−0.019** (0.005)	**−10.017** (1.525)
36 months	**2.444** (0.119)	**3.534** (0.140)	**−0.012** (0.001)	**−5.484** (0.598)

Horizon	(c) short rate	10y - 1m spread	lagged growth	10y infl. prem.
12 months	4.072 (4.194)	4.056 (3.396)	−0.013 (0.012)	13.235 (10.265)
24 months	**5.455** (1.208)	**5.042** (1.035)	**−0.019** (0.004)	**15.545** (2.386)
36 months	**3.230** (0.178)	**3.341** (0.128)	**−0.012** (0.001)	**8.535** (0.952)

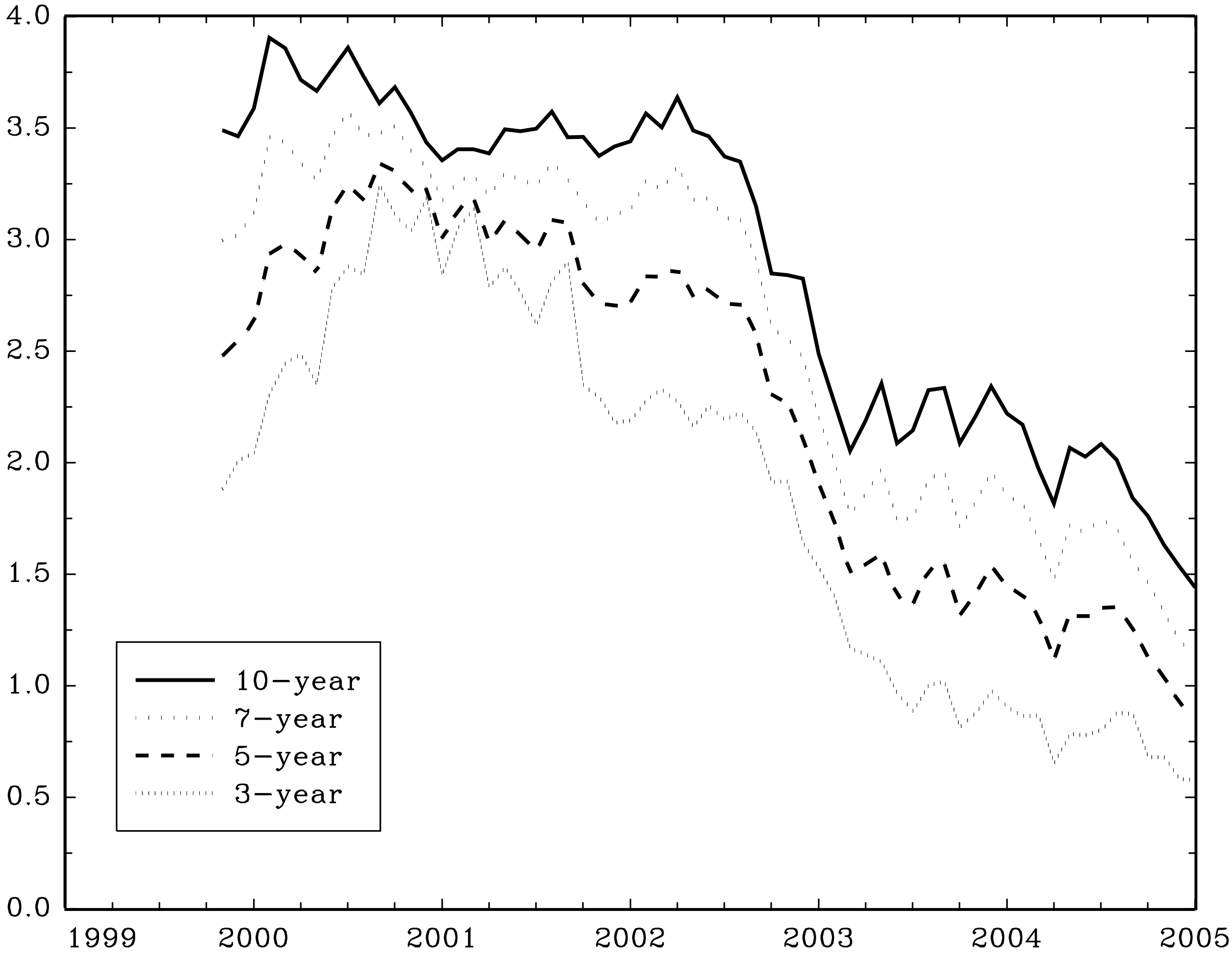
Figures in parentheses are Hodrick (1992) standard errors. Bold figures denote significance at the 5 percent level. The regressions are run over the entire 1991-2004 sample.

Table 4: Parameter estimates for predictive inflation regressions

	(a)			
Horizon	short rate	10y - 1m spread	lagged inflation	10y. break-even
12 months	−0.245 (0.375)	−1.306 (0.755)	−0.008 (0.016)	4.706 (2.983)
24 months	**−0.618** (0.149)	**−1.972** (0.230)	**−0.024** (0.006)	**7.451** (0.951)
36 months	**−0.777** (0.097)	**−2.105** (0.248)	**−0.021** (0.004)	**7.835** (1.049)
	(b)			
Horizon	short rate	10y - 1m spread	lagged inflation	corr. break-even
12 months	−0.689 (0.646)	−1.204 (0.690)	−0.001 (0.012)	1.693 (1.069)
24 months	**−1.315** (0.234)	**−1.809** (0.207)	−0.012 (0.007)	**2.679** (0.338)
36 months	**−1.507** (0.200)	**−1.930** (0.225)	**−0.009** (0.003)	**2.811** (0.376)
	(c)			
Horizon	short rate	10y - 1m spread	lagged inflation	10y infl. prem.
12 months	−0.936 (0.799)	−1.145 (0.654)	0.003 (0.011)	−2.641 (1.665)
24 months	**−1.705** (0.281)	**−1.716** (0.195)	−0.006 (0.008)	**−4.180** (0.524)
36 months	**−1.914** (0.256)	**−1.831** (0.212)	−0.003 (0.005)	**−4.380** (0.586)

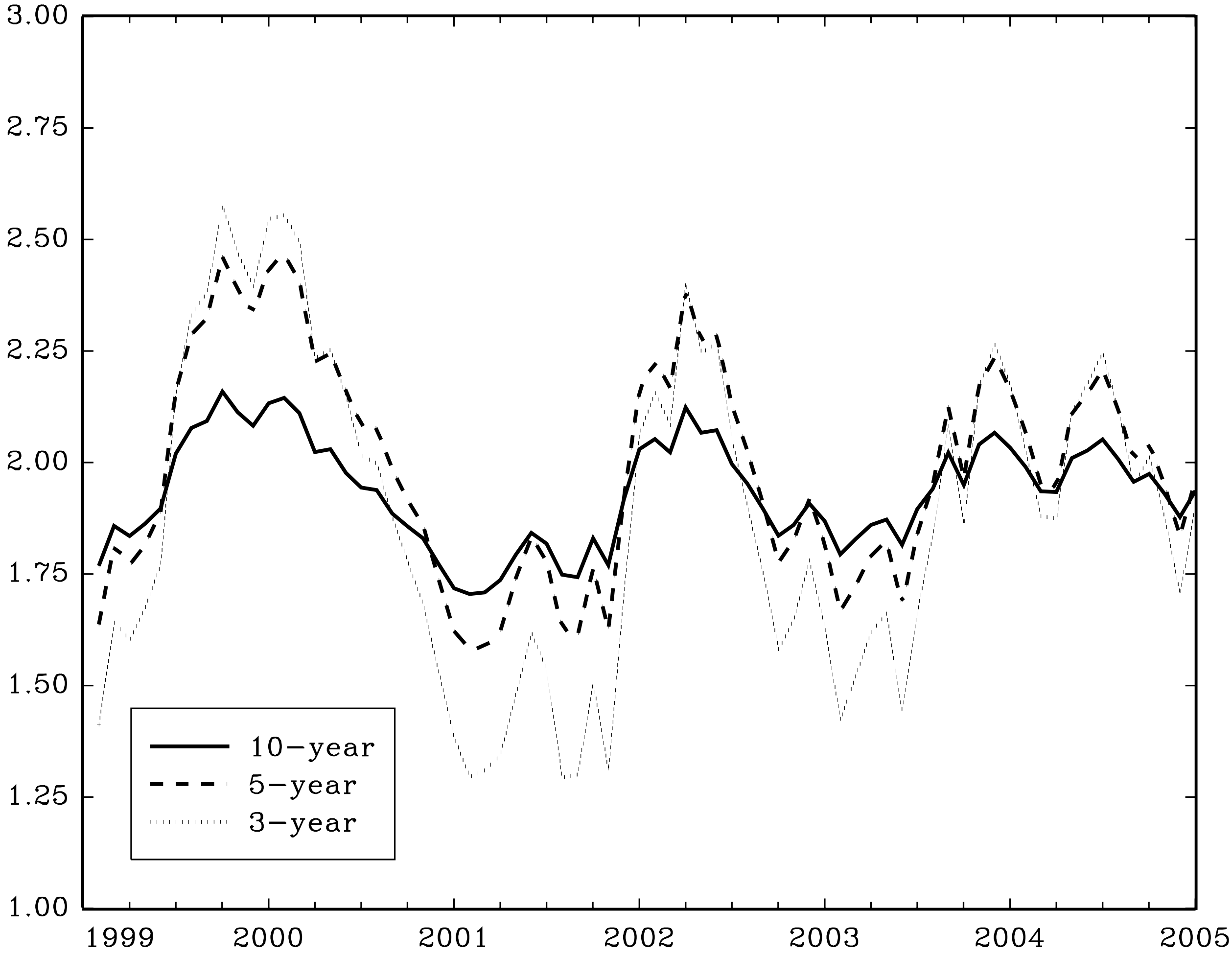
Figures in parentheses are Hodrick (1992) standard errors. Bold figures denote significance at the 5 percent level. The regressions are run over the entire 1991-2004 sample.

Figure 1: Real zero-coupon yields



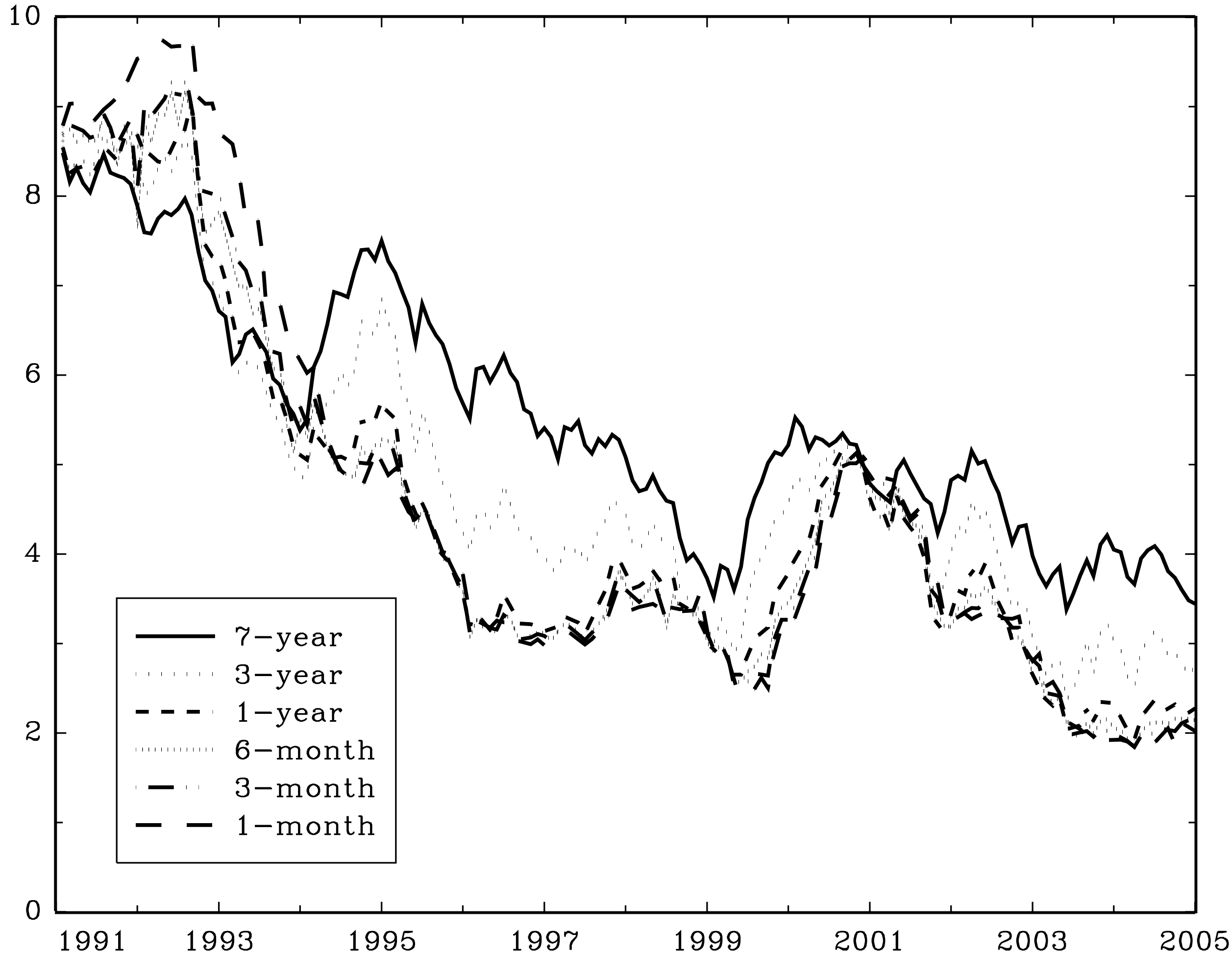
Percent per year.

Figure 2: Break-even inflation rates



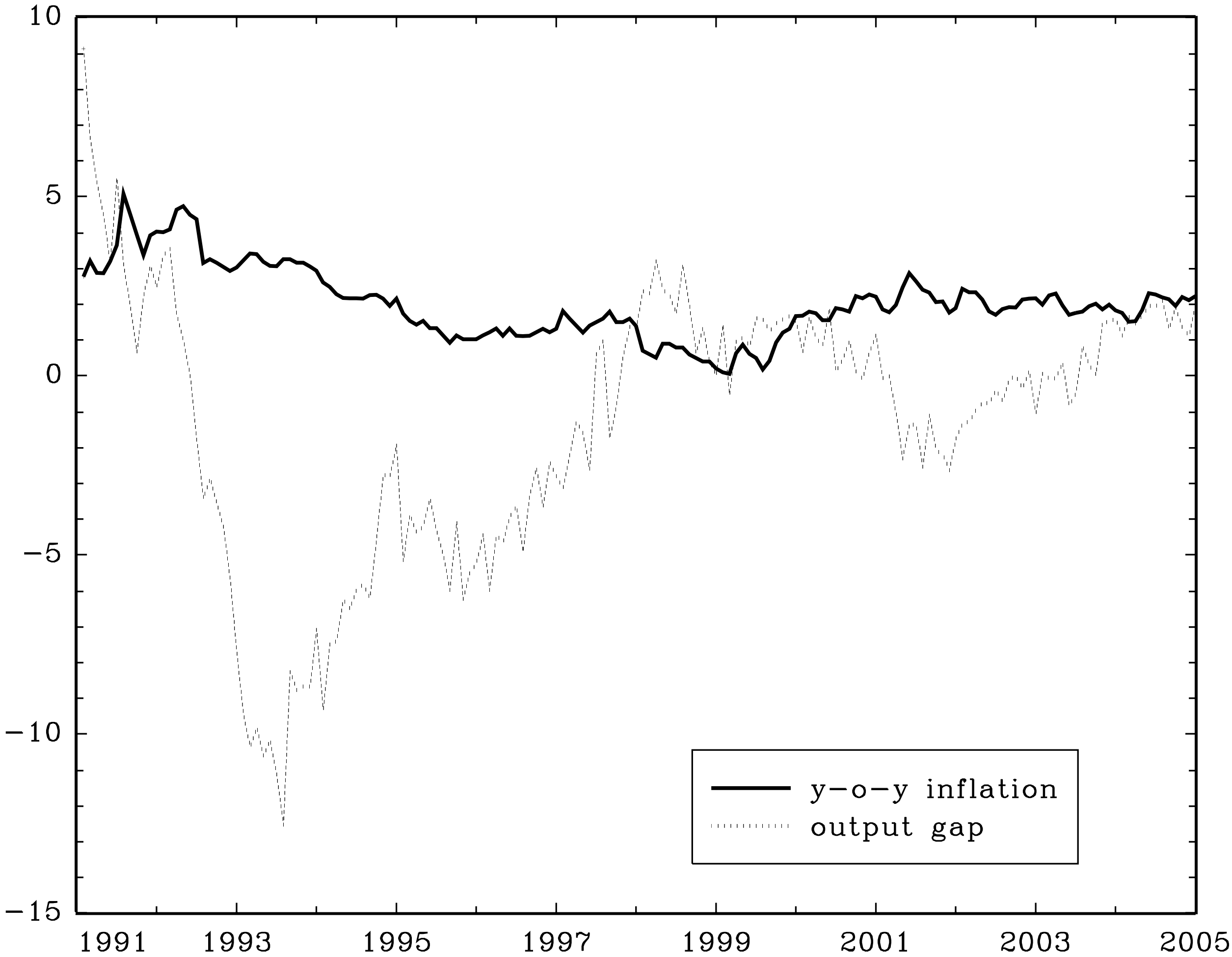
Nominal minus real zero-coupon yields; percent per year.

Figure 3a: Nominal zero-coupon yields



Percent per year.

Figure 3b: Inflation and output gap



Annualized percentages.

Figure 4a, b: Impulse responses to a macro shocks

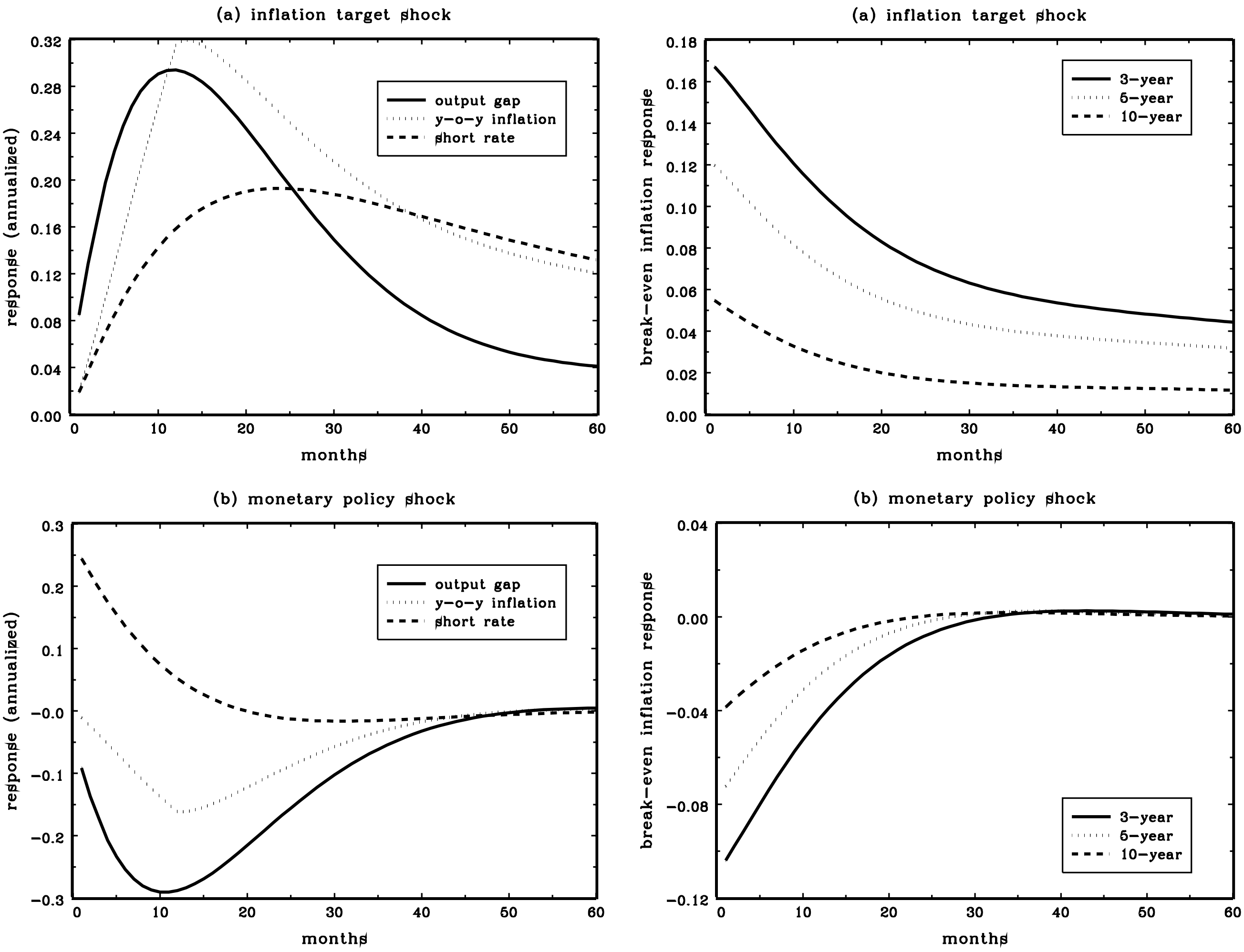


Figure 4c, d: Impulse responses to a macro shocks

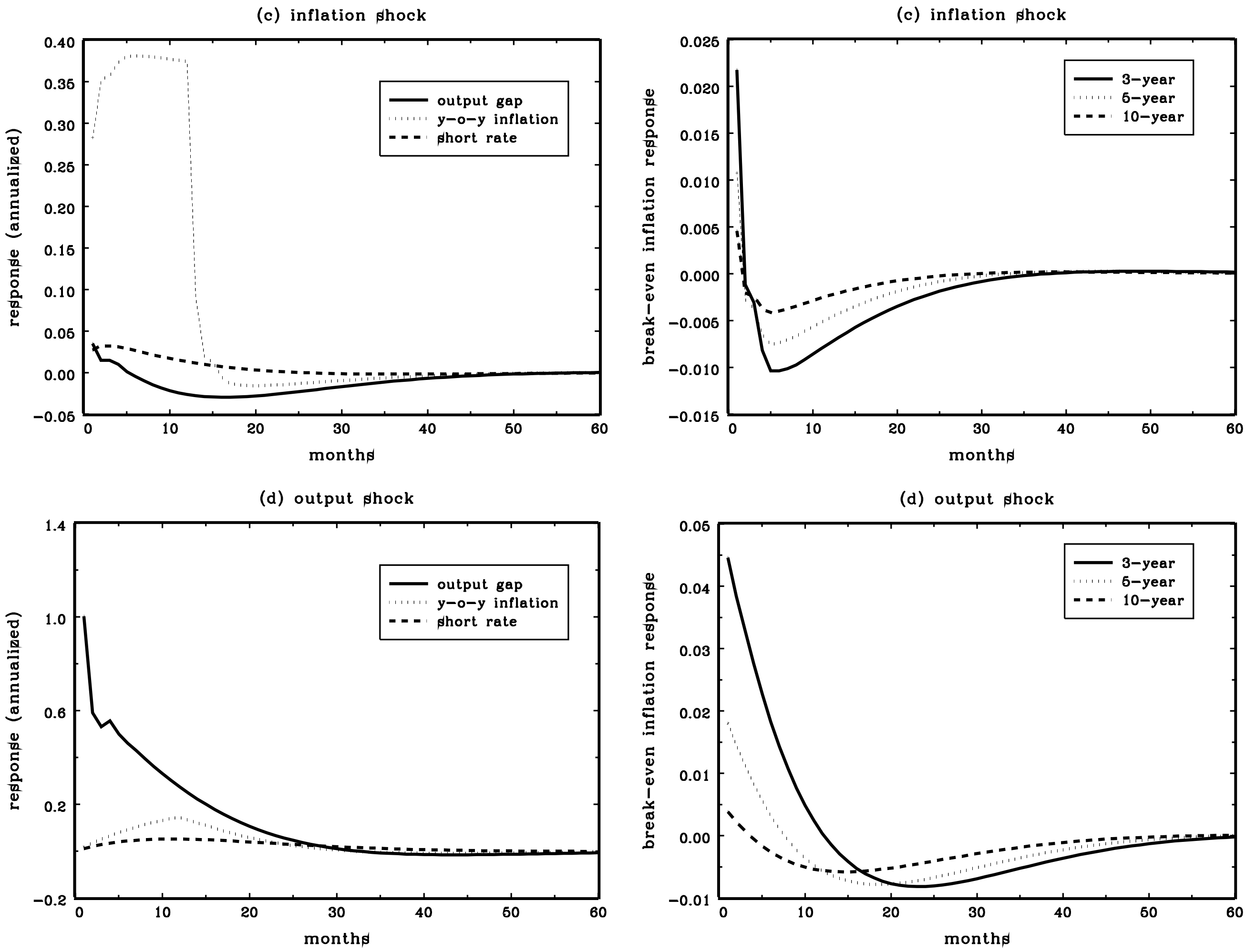


Figure 5: Historical decomposition of the nominal 3-month interest rate

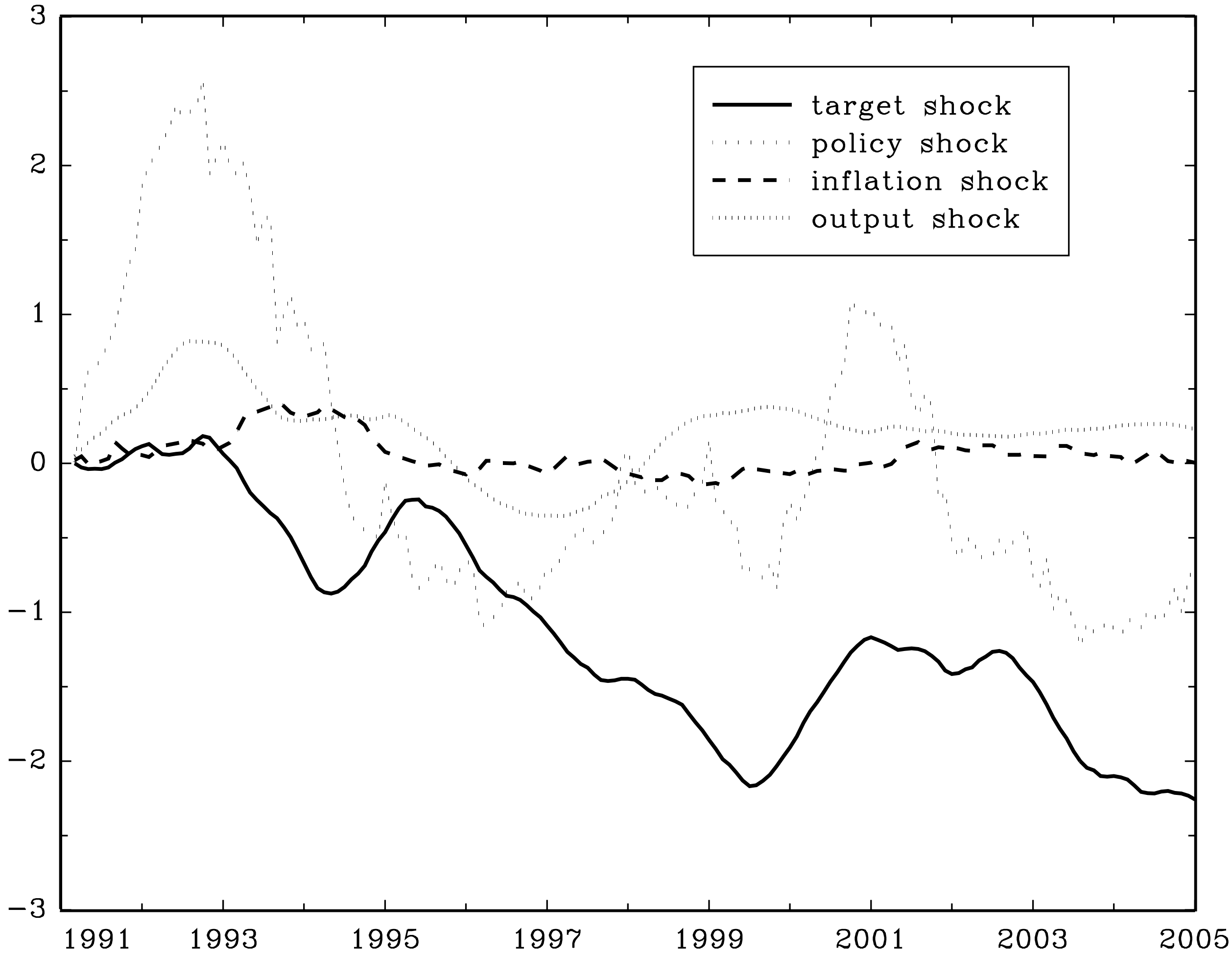
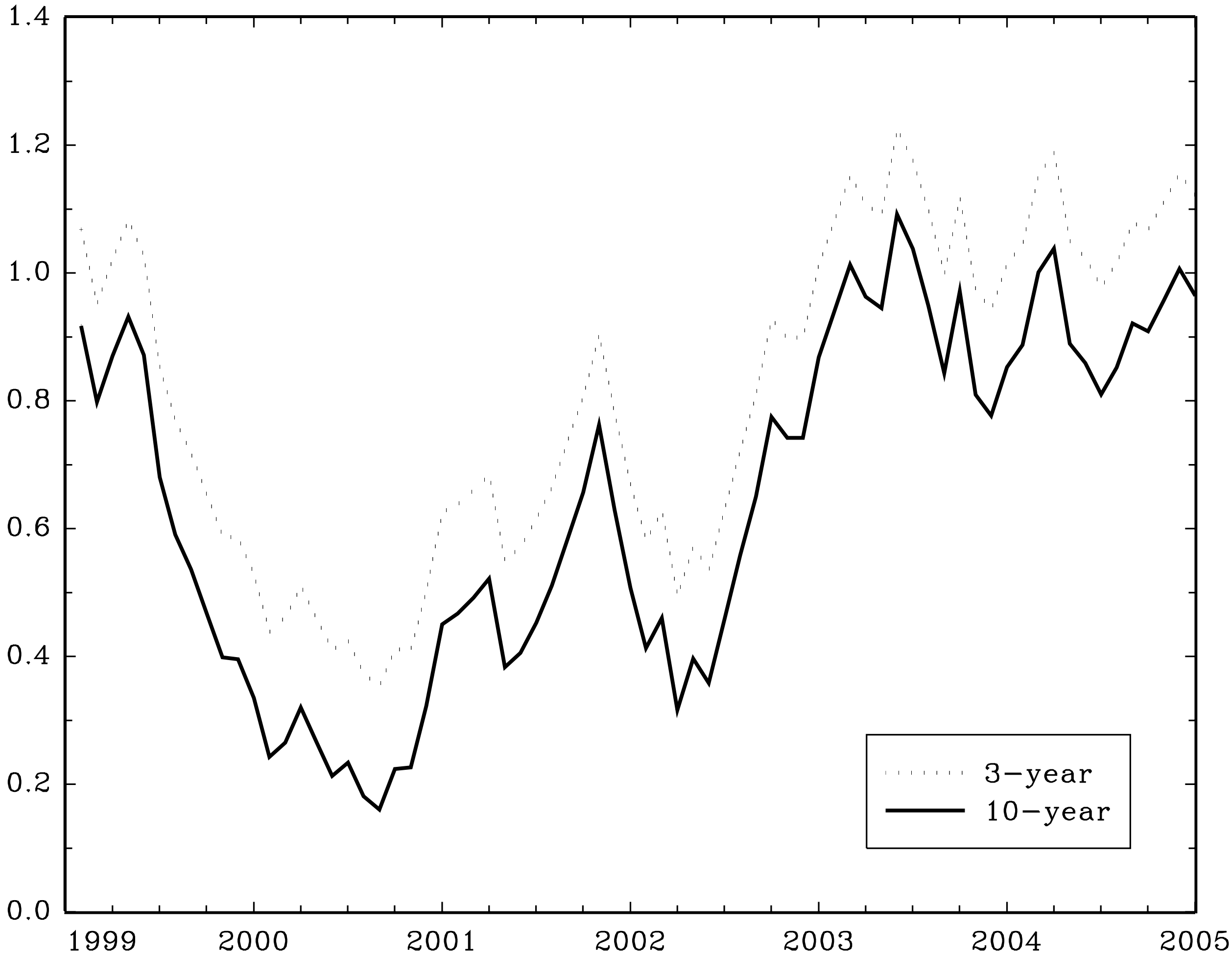


Figure 6a: Inflation risk premia



Percent per year.

Figure 6b: Decomposition of 10-year inflation risk premium

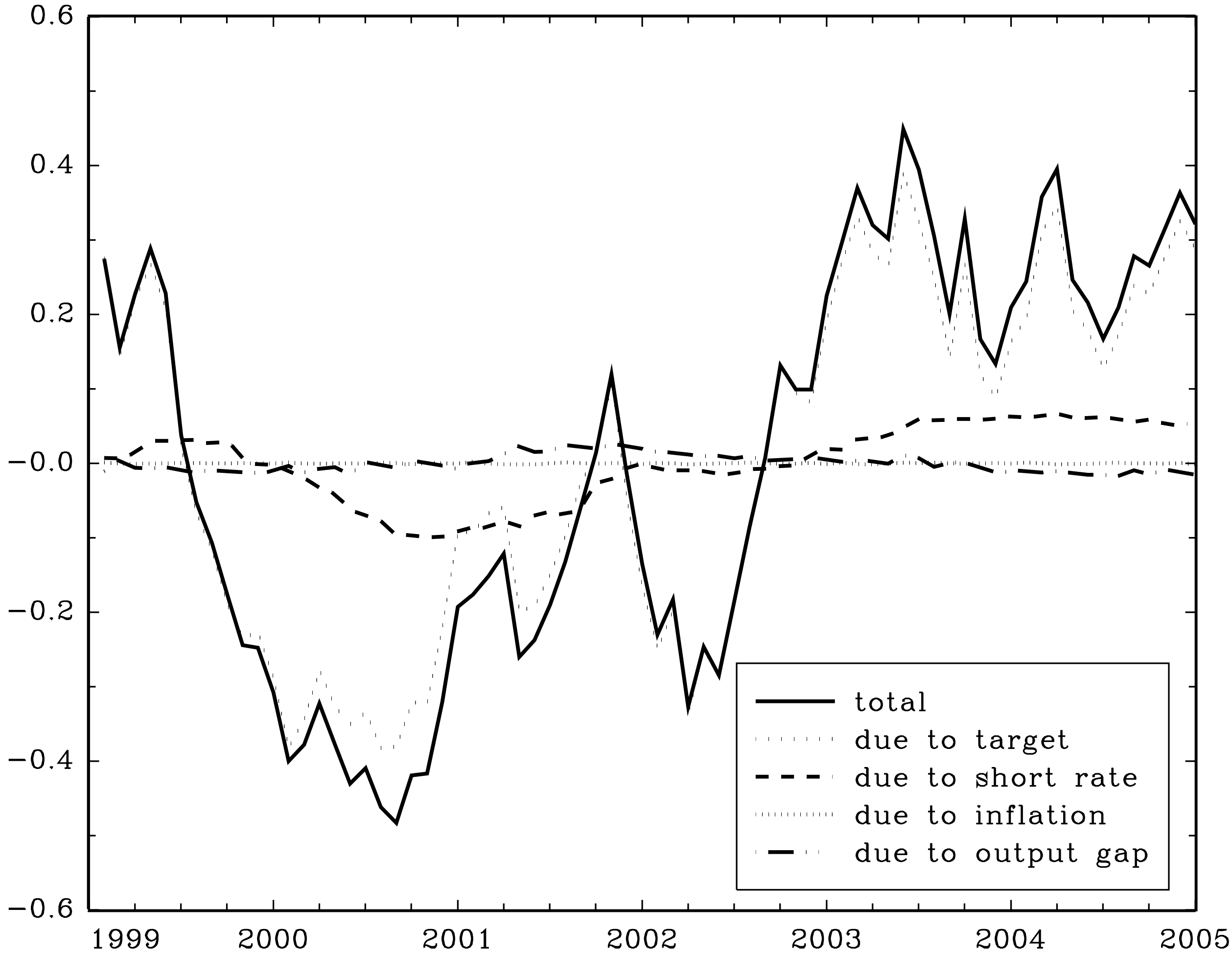


Figure 7: Risk-adjusted 10-year break-even inflation and perceived inflation objective