Solving Models of Optimal Monetary and Fiscal Policy by Projection Methods

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Abstract

The paper presents some methods that allow to compute approximations to optimal Ramsey problems that are more precise than the solutions usually found in the literature. The methods are illustrated with a recent sticky-price model of optimal monetary and fiscal policy under commitment. The paper shows how to compute higher order approximations in a neighbourhood of the steady state curve (the set of all deterministic states, not just one of them). and also how to compute the stochastic solution around a full transition path of the Ramsey policy. The methods should be applicable to a wide range of models of optimal fiscal and monetary policy.

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1 Introduction

Important progress has been made in recent years on the theory of optimal Ramsey taxation. A number of papers (e.g., Benigno and Woodford, 2003;Schmitt-Grohé and Uribe, 2004a; Schmitt-Grohé and Uribe, 2005;) investigates joint optimal monetary and fiscal policy under incomplete markets and price rigidities, which provides an interesting and realistic framework. These models cannot be solved exactly by analytical means, and even the numerical solution raises some important technical problems. The existing literature usually solves the model by local approximations of first or second order. The aim of this paper is to provide a set of tools that allow higher order global approximations to the solution. The basic technique used are projection methods (Judd 1998, Ch. 11). To apply them successfully, it is necessary to choose carefully the state space on which to approximate the solution.

To illustrate the working of the methods, I apply them here to a prominent model of the recent literature (Schmitt-Grohé and Uribe 2004a). But the techniques applied are quite general, and suitable for an important class of models.

Contrary to many models that can be satisfactorily solved by local approximation techniques, a full solution to models of optimal fiscal and monetary policy requires global approximation techniques for two reasons. First, optimal Ramsey policies with commitment usually do not have a time invariant solution in the natural state space. To make them recursive, the state space has to be enlarged by some costate variables, typically the Lagrange multipliers of the dynamic household first order conditions. In the first period of the problem, the value of the costates is equal to 0 (Marcet and Marimon 1998). In the starting period, we will in general be far away from a steady state, so that a local approximation around some steady state cannot be expected to give a reasonable approximation to the general problem, starting in period 0. Second, problems of optimal fiscal policy with government debt usually do not have a unique deterministic steady state. The level of government debt in steady state depends in general on the initial condition and the full transition path to the steady state. These models then have a continuum of deterministic steady states, which can be indexed by the level of government debt. This means that over time the solution of the stochastic model will wander far away from any given steady state, so that a local approximation around one of the steady states will probably become very inaccurate after some time.

With respect to the first problem, the literature usually confines itself to computing the stochastic steady state policy, which is the often called the “optimal policy from a timeless perspective” (Woodford 1999). This is obviously a restriction, since what we get is only a part of the full solution. With respect to the second problem, the literature only considers simulations of the model with a limited time horizon, such that the model solution probably stays close to the steady state about which the approximation was taken. This is good enough in many applications, but it can hardly be recommended as a general procedure, and it is often difficult to check whether the obtained solution is accurate enough for the specific purpose.

address some economic questions that need high-precision solution

1.1 Computer programs

The programs to do the computations in this paper need some symbolic differentiation, for which i use MuPAD, a free computer algebra system available at http:www.mupad.de. Most of the code is written for FastMat, which is a new matrix oriented language written by myself, with a syntax that is highly compatible to Matlab, but has some useful extras (apart from, obviously, not yet having many things that Matlab has). Fastmat is in general faster than Matlab. Right now, it is not yet ready for public use, but will be made available early next year. Some of the programs can be used with Matlab after small modifications. To compute the log-linear approximations, I adapted routines from Uhlig's toolkit (Uhlig 1998) and from the CompEcon toolkit of (Miranda and Fackler 2002). The programs will be in nice shape and ready for public use later this year (November or December). If you are interested in obtaining programs, please email me at michael.reiter@upf.edu.

2 The model

The model that I investigate here is taken from Schmitt-Grohé and Uribe (2004a). I will therefore describe the model only very briefly. A more detailed description and discussion can be found in the original source.

2.1 The policy problem

The basic question that the model wants to answer is how a government should optimal react to spending and technology shocks. Basically, the government has three options: varying taxes, varying the inflation rate and varying government debt. Varying taxes is bad, because excess burden is a convex function of taxes, so we prefer smooth marginal tax rates. By creating surprise inflation, the government can pass on shocks to the holders of nominal bonds. The resulting inflation variability is costly if there is some degree of prices stickiness. Varying government debt obviously has its limits.

In a model without monetary policy, Barro (1979) has argued that government debt should be used to smooth tax rates over time, and that optimal debt and taxes approximately follow a random walk. Marcet and Scott (2003) find that this is a good approximation for US policy. However, Lucas and Stokey (1983) address the same question in a model with complete markets, and show that the Barro result does not hold: optimal taxes and debt are not random walks, but inherit the stochastic properties of the shocks. In a world of complete markets, the government has enough instruments to insure against shocks, and need not use debt for that purpose. Aiyagari, Marcet, Sargent and Seppälä (2002) show that the complete markets assumption is essential: if markets are incomplete in the sense that the only debt instrument that the government has is riskless real debt, than we are approximately back at the Barro result.

The situation changes once we allow for *nominal* government debt. When prices are flexible, unexpected inflation turns out to be a good shock absorber, passing on the burden to the holders of nominal debt, such that the real allocation comes very close to the allocation under perfect markets (G.A. Calvo, 1990, G.A. Calvo, 1993 V.V. Chari, 1991, Schmitt-Grohé and Uribe, 2004a). The question then is whether this result holds up under some form of price stickiness. In that case, varying inflation is costly, and it is a priori not clear in which direction the tradeoff between varying taxes or varying inflation is solved. The conclusion of Schmitt-Grohé and Uribe (2004a) is that it is unambiguously solved toward keeping inflation stable and varying taxes (although it has to be modified somewhat when wages are sticky, cf. for example Schmitt-Grohé and Uribe, 2005).

2.2 Long-run debt dynamics

The random-walk property of debt is (if at all) only approximately true. The theoretical analysis of a simpler model (Aiyagari et al. 2002) shows that in the very long run, the Ramsey government accumulates enough assets to finance government expenditures by interest revenues. One could call this "government precautionary savings": very high levels of debt force the government to adopt very high marginal tax rates with an ever increasing excess burden, and from some point on, the government may even be unable to serve the debt. Since adverse spending or technology shocks drive us in this direction, a prudent government wants to stay away from this dangerous region and steadily decrease its debt level (build up assets). How important is government precautionary savings quantitatively? At current debt levels, how much would optimal debt reduction on average be? Quantifying precautionary saving is rather subtle. It is clear that one needs a global approximation to answer the question reliably, and the solution probably needs to be quite accurate. In the following I will try to see how far we can go in this direction.

2.3 The model: outline

- Households choose consumption, leisure, and money
- Money reduces the transaction costs of consumption purchases
- Firms: imperfect competition, price adjustment costs
- Government solves Ramsey problem under full commitment, government expenditures exogenous
- Instruments:
  - Tax rate on labor income
  - Nominal debt, not state contingent
  - Money supply
- AR(1) processes for technology and government expenditures

2.4 The Household/Firm

The economy is populated by a representative household/firm with infinite horizon. It maximizes discounted expected utility $\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, h_t)$ where

$$U(c, h) = \ln(c) + \delta \ln(1 - h) \tag{1}$$

where c is Dixit-Stiglitz aggregate of intermediate consumption goods, and h is labor input.

Each household owns an intermediate goods firm. The firm works with linear technology (a tilde always characterizes variables relating to the firm):

$$Y_t = z_t \tilde{h}_t \tag{2}$$

where productivity z_t is an exogenous productivity parameter that is common to all firms and follows the AR(1) process

$$\ln z_t = \lambda^z \ln z_{t-1} + \epsilon_t^z, \qquad \epsilon_t^z \sim N(0, \sigma_{\epsilon^z}^2) \tag{3}$$

Money demand is motivated by transaction costs in consumption purchases. Define the consumption velocity of money as

$$\nu_t \equiv P_t c_t / M_t \tag{4}$$

Greater real money holdings (reduction in ν) save transaction costs, which we assume are of the following form:

$$s(\nu) = A\nu + B/\nu - 2\sqrt{AB} \tag{5}$$

Price stickiness is introduced into the model by a quadratic price adjustment cost function following Rotemberg (1982):

$$PAC_t = \frac{\theta}{2}\left(\frac{\tilde{P}_t}{\tilde{P}_{t-1}} - 1\right)^2 \tag{6}$$

From the properties of a Dixit-Stiglitz aggregator it is well know that the demand for a firm's product can be written as $Yd\left(\tilde{P}_t/P_t\right)$. Each firm then yields the following cash flow:

$$CF_t = \frac{\tilde{P}_t}{P_t} Y_t d\left(\frac{\tilde{P}_t}{P_t}\right) - w_t \tilde{h}_t \tag{7}$$

Bonds are one-period and nominally risk free. Define R_t as the risk-free nominal interest factor, such that a dollar of dividends in $t+1$ costs R_t^{-1} dollars in t. The budget constraint of the household/firm is then

$$P_t c_t \left(1 + s(\nu_t)\right) + M_t + R_t^{-1} D_{t+1} = \\ M_{t-1} + D_t + P_t \left[CF_t - PAC_t\right] + (1 - \tau_t) P_t w_t h_t \tag{8}$$

Since in symmetric equilibrium there is no bond trade among households, we have $D_t = R_{t-1} B_{t-1}$.

2.5 The government

Government expenditures g_t are exogenous, and unproductive following the AR(1) process

$$\ln g_t = (1 - \lambda^g)\bar{g} + \lambda^g \ln g_{t-1} + \epsilon_t^g, \qquad \epsilon_t^g \sim N(0, \sigma_{\epsilon^g}^2) \tag{9}$$

The are financed by

- a labor income tax at rate τ_t
- one-period debt B_t, which is nominal and non-contingent (*real* debt is then state-contingent)
- money M_t

The government budget constraint is

$$M_t + B_t = M_{t-1} + R_{t-1} B_{t-1} + P_t (g_t - \tau_t w_t h_t) \tag{10}$$

2.6 The government Lagrangian

The Lagrangian of the government problem is (Schmitt-Grohé and Uribe 2004a, p.223)

$$\begin{aligned}\mathcal{L} = {} & \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, h_t) + \lambda_t^f \left[z_t h_t - (1 + s(\nu_t)) c_t - g_t - \frac{\theta}{2}(\pi_t - 1)^2\right] \\ & + \lambda_t^b \left[\lambda_t - \beta \rho(\nu_t) \,\mathrm{E}_t \left(\lambda_{t+1}/\pi_{t+1}\right)\right] \\ & + \lambda_t^s \left[\frac{c_t}{\nu_t} + b_t + \left(mc_t z_t + \frac{U_h(c_t, h_t)\gamma(\nu_t)}{U_c(c_t, h_t)}\right) h_t - \frac{\rho\left(\nu_{t-1} b_{t-1}\right)}{\pi_t} - \frac{c_{t-1}}{\nu_{t-1}\pi_t} - g_t\right] \\ & + \lambda_t^p \left[\beta \,\mathrm{E}_t \frac{\lambda_{t+1}}{\lambda_t} \pi_{t+1} \left(\pi_{t+1} - 1\right) + \frac{\eta z_t h_t}{\theta}\left(\frac{1+\eta}{\eta} - mc_t\right) - \pi_t \left(\pi_t - 1\right)\right] \\ & + \lambda_t^c \left[U_c(c_t, h_t) - \lambda_t \gamma(\nu_t)\right]\end{aligned} \tag{11}$$

where

$$\begin{aligned}
\gamma(\nu) &\equiv 1 + s(\nu) + \nu * s'(\nu) \\
\rho(\nu) &\equiv 1/(1 - \nu^2 * s'(\nu)) \\
mc_t &\equiv w_t/z_t \\
b_t &\equiv B_t/P_t \\
\pi_t &\equiv P_t/P_{t-1}
\end{aligned}$$

Here mc_t stands for "marginal costs".

2.7 State variables

For a nonlinear approximations of the model, it is crucial to reduce the number of state variables as much as possible. It turns out that we need 5 state variables:

- Endogenous
  - Real government debt + money:

    $$a_{t-1} \equiv (M_{t-1} + R_{t-1}B_{t-1})/P_{t-1} \tag{12}$$

    We see from the household budget constraint (8) and the government budget constraint (10) that M_{t-1} and B_{t-1} only enter in the combination $M_{t-1} + R_{t-1}B_{t-1}$, so the two can be aggregated into one state variable a_{t-1} after dividing equations by the price level.
  - Costate: lagrange multiplier of HH euler equation for bonds λ^b
  - Costate: lagrange multiplier of HH euler equation for prices λ^p
- Exogenous
  - Productivity z_t
  - Government expenditures g_t

It is also useful to reduce the number of control variables. I found it convenient to work with the variables c, mc, Pi, ν and λ^s and eliminating other variables by the following definitions:

$$\begin{aligned}
h_t &\equiv ((1 + s(\nu_t))c_t + g_t + \frac{\theta}{2}(Pi_t - 1)^2)/z_t \\
\lambda_t &\equiv U_c(c_t, h_t)/\left(1 + s(\nu_t) + \nu_t s'(\nu_t)\right) \\
R_t &\equiv 1/(1 - \nu_t^2 s'(\nu_t)) \\
\tau_t &\equiv 1 + U_h(c_t, h_t)/(\lambda_t z_t mc_t) \\
m_t &\equiv c_t/\nu_t \\
b_t &\equiv (a_t - m_t)/R_t
\end{aligned}$$

which are either definitions or can be solved for from static first order conditions. Expressing the Lagrangian in this reduced set of variables, the first order conditions become very lengthy. The only reasonable way to proceed is to use a computer algebra system to derive symbolically the first derivatives of the Lagrangian, and put the resulting equations in the program. Good algebra systems have an option to produced "optimized code", that means, it computes the equations efficiently by computing and storing appropriate intermediate results. The output will be in either C or Fortran language, but porting this into Matlab syntax is trivial.

2.8 Calibration

Following Schmitt-Grohé and Uribe (2004a, Section 4.1), these are the parameter values used:

Parameter	Values
β	1/1.04
η	-6
θ	4.375
δ	2.9
A	0.0111
B	0.07524
$\bar{g}$	0.04
λ^g	0.9
λ^z	0.82
$\sigma^2_{\epsilon^g}$	0.0302
$\sigma^2_{\epsilon^z}$	0.0229

The time period of the model is a year. The parameter for η implies a steady state markup of 20%. In the benchmark steady state (about which Schmitt-Grohé and Uribe (2004a) approximate), production is 0.2, government debt (excluding money) is 44% of production, and government consumption is also 20% of production on average ($\bar{g}$). The consumption velocity of money turns out to be about 3.18, so that total real debt (including money) is about 70% of annual GDP. Price stickiness parameters follow the estimates in Sbordone (2002). For a more detailed justification of the calibration, see Schmitt-Grohé and Uribe (2004a, Section 4.1).

3 The deterministic model

3.1 Steady states

The results show that some variables are almost constant across steady states. In the stochastic model, we can expect those variables to be stationary. Among them are inflation and money velocity.

Other variables, mainly the labor tax rate, income, labor input and consumption, vary substantially across steady states and will appear non-stationary in the simulations of the stochastic model.

What these results show again is that the inflation tax is a very poor way of making money in the steady state: the Ramsey government prefers raising the tax rate to 40% rather than increasing inflation beyond its level of about -0.1%.

3.2 Transition paths in the deterministic model

A very useful intermediate step in the analysis is to solve the deterministic version of the model. From this we first learn how fast the solution converges to a steady state, to what is often called the "timeless perspective". If we converge very quickly, it lends some support to the idea that "policy from a timeless perspective" is the right concept for practical policy purposes. We will see that this is in fact the case for the current model. Second, by solving the model starting from many different starting points, we can trace out the relevant part of the state space, which is essential for obtaining an accurate solution of the model (cf. Section 4.1).

There are many techniques to compute the solution of the deterministic model (Judd 1998, Ch. 16). My favourite is the most direct one: solve for all the equations of the model simultaneously, for a finite time horizon that is long enough to reach a steady

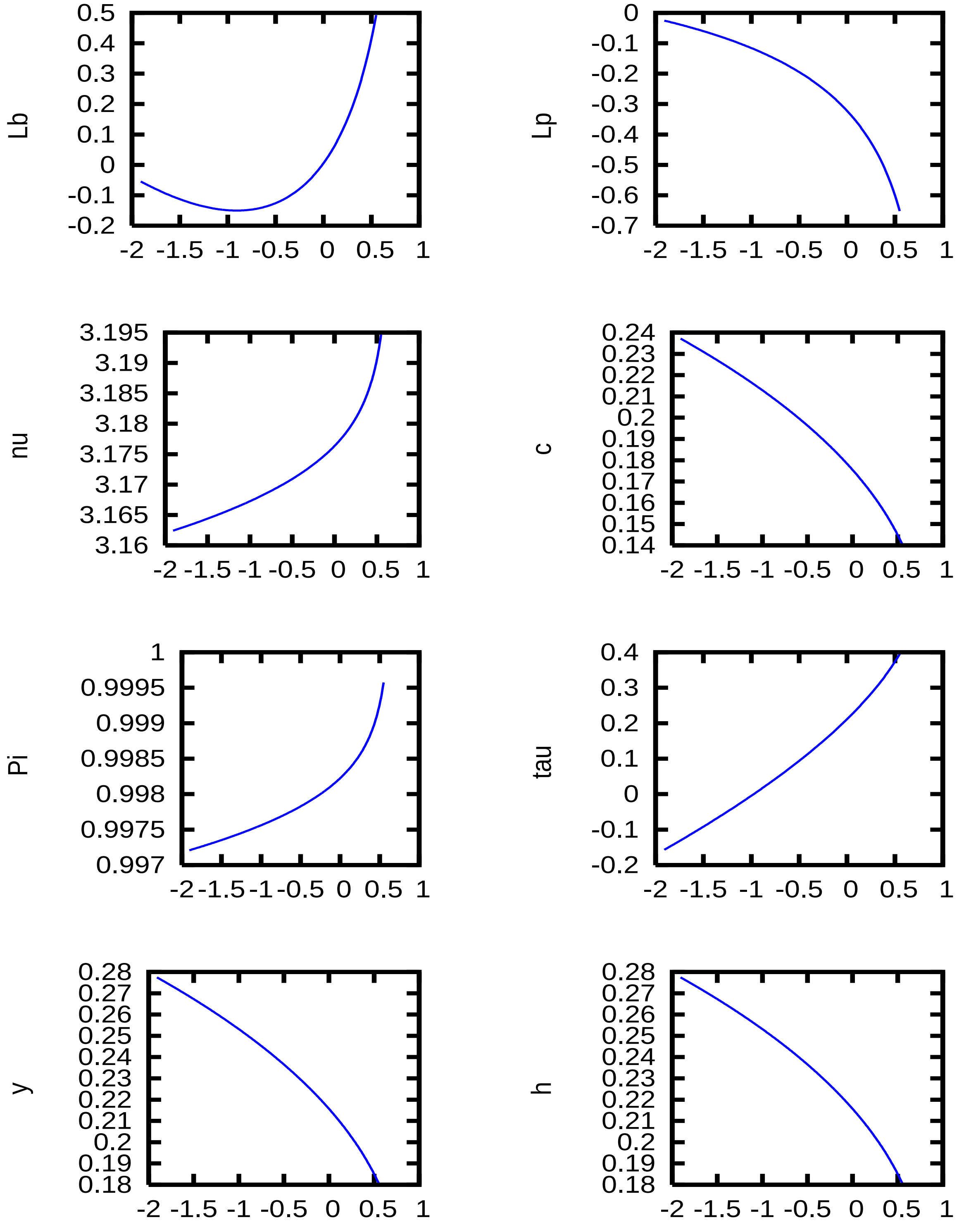

Figure 1: Results for different steady states; x-axis: real debt+money, a

state, such that a steady state condition can be imposed at the end of the finite horizon. In our model, what I do concretely is the following. For the 8 model variables a, λ^b, λ^p, c, mc, Pi, ν and λ^s, we get in each period 8 first order conditions from the derivatives of the Lagrangian (11).

Given are a_0 (arbitrary starting point), $\lambda_0^b = 0$ and $\lambda_0^p = 0$. That the costate variables start at 0 has the following interpretation (Marcet and Marimon 1998). The costates measure the effect of the earlier commitment of the planner on the current solution. In the first period, there are no earlier binding commitments that the planner has to follow, therefore the Lagrange multipliers are 0.

Choosing a time horizon of T, I solve a problem of $8T + 3$ equations in $8T + 3$ unknowns. The unknowns are

- a_t, λ_t^b, λ_t^p, c_t, mc_t, Π_t, ν_t and λ_t^s for $t = 1, \ldots, T$.
- ν_{T+1}, c_{T+1} and Π_{T+1}

The equations are

- The derivative of (11) w.r.t. a_t, λ_t^b, λ_t^p, c_t, mc_t, Π_t, ν_t and λ_t^s for $t = 1, \ldots, T$.
- The derivative of (11) w.r.t. c_{T+1}, Π_{T+1} and λ_{T+1}^s.

I choose $T = 200$, which is far more than enough to come close to a steady state. This results in 1603 equations in 1603 unknowns, which may look like a formidable computational problem. However, the Jacobian of the equation system is very sparse, since in any equation only a few different variables enter. To solve this system efficiently[1], one therefore needs

- an automatic differentiation tool that takes the sparsity into account, so that the Jacobian can be computed quickly
- an efficient way to solve the linear equation for the Newton step, taking into account the sparsity of the Jacobian. It turns out that *direct sparse methods* are much better than iterative methods such as GMRES. A very good non-commercial package that does this is UMFPACK (currently version 4.4), which is also used by Matlab (in this application, it proved to be about twice as fast as SuperLU, which is another free package).

In the first periods, the planner is not yet bound by earlier promises, and can use some "outrageous" policies to raise a lot of money to reduce the public debt burden, exploiting quasi-fixed factors as much as possible. In our model, the central planner wants to choose a very high inflation rate in the first period, thereby expropriating the owners of nominal government debt. Although in the current model the inflation rate would not go to infinity in the first period, due to the quadratic price adjustment costs, inflation would still be unrealistically high. I therefore follow the usual practice and impose a (basically arbitrary) restriction on inflation, namely an upper bound of 10%. With this inequality constraint, the problem is not a pure root-finding problem any longer, but has a complementarity subproblem. I convert the problem back into a root-finding problem by smoothing the complementarity part (more concretely, the derivative of the Lagrangian w.r.t Π_t) using Fischer's equation (Miranda and Fackler 2002, Section 3.8).

3.3 Results

Figure 2 displays all the combinations of a_t, λ_t^b and λ_t^p that were obtained in the solution of the deterministic model, starting from a wide range of values for a_0, the initial level of

[1] With 1603 equations, even a naive interpretation in Matlab, computing the Jacobian by forward differencing and ignoring the sparseness is quite feasible.

real public debt (including money). The Lagrange multipliers are always set to 0 in the first period, which is what theory commands. Figures 3 and 4 show the projections into (a, λ^b)- and (a, λ^p)-space, respectively. In each case, the straight line that is separate from the other points represents the different starting values (with the multipliers set to zero).

The results in Figures 2 to 4 are interesting for at least two reasons. First, they show us that the optimal solution converges to one of the steady states very quickly. From most starting points, we are very close to a steady state already in the second period. Only when we start from very high debt levels, there is a more protracted transition period. With very high debt levels, the government chooses a high inflation rate for several periods, as can be seen from Figure 5. The reduction in the real value of the debt is then more important than the price adjustment costs that results from high inflation. As mentioned above, the quick convergence to a steady state makes it even more plausible that a policy maker should act like being in a steady state ("policy from a timeless perspective").

What these results also show is that, apart from the very first period, the relevant state space in which the solution of the model lives is rather small. This is something we have to take into account when designing a global approximation to the solution.

4 Global stochastic approximations

4.1 Methods

Recently, perturbation methods have become a popular tool (Collard and Juillard, 2001; Jin and Judd, 2002; Chen and Zadrozny, 2003; Schmitt-Grohé and Uribe, 2004b) to compute higher-order approximations for many types of models. Perturbations provide local higher-order approximations about some object like a steady state or a path. Most applications only apply a perturbation around a deterministic steady state, although Judd (1998, Part IV) makes clear that the method is much more general.

For computing global approximations, the obvious tool are projection methods (cf. Judd, 1992 or Judd, 1998, Ch. 11). What I want to stress here is that projection methods are extremely flexible. Besides global solutions, they also allow to get local solutions around a given steady state (or set of steady states, or a deterministic path, as shown below), simply by approximating the relevant functions on a very small state space around those objects. Applied in this way, projection methods can be used to obtain the same approximations that perturbation methods give. Of course, there is the issue of which method is computationally more efficient. A good implementation of perturbation methods should be more efficient than a projection method to obtain local approximations, although I am not sure that the difference would be big, for the following reason. When approximating a function on a very small grid, the projection method converges in very few Quasi-Newton steps, whenever one has starting values from a locally valid approximation (for example a linear approximation, as explained in Section 4.1.6). Experience shows that the Jacobian of the residuals has to be computed only once in this case.f If the projection method is well implemented, making use of automatic differentiation (only Jacobians needed!) just as perturbation methods do, it may be quite efficient.

Projection methods give us the freedom to choose the state space on which we want to approximate the solution, which can be a narrow or a large region around a steady state, a set of steady states, a deterministic path etc. We can choose big noise or small noise. In other words, one set of tools allows a wide variety of solutions, and this is what I will explore in the following.

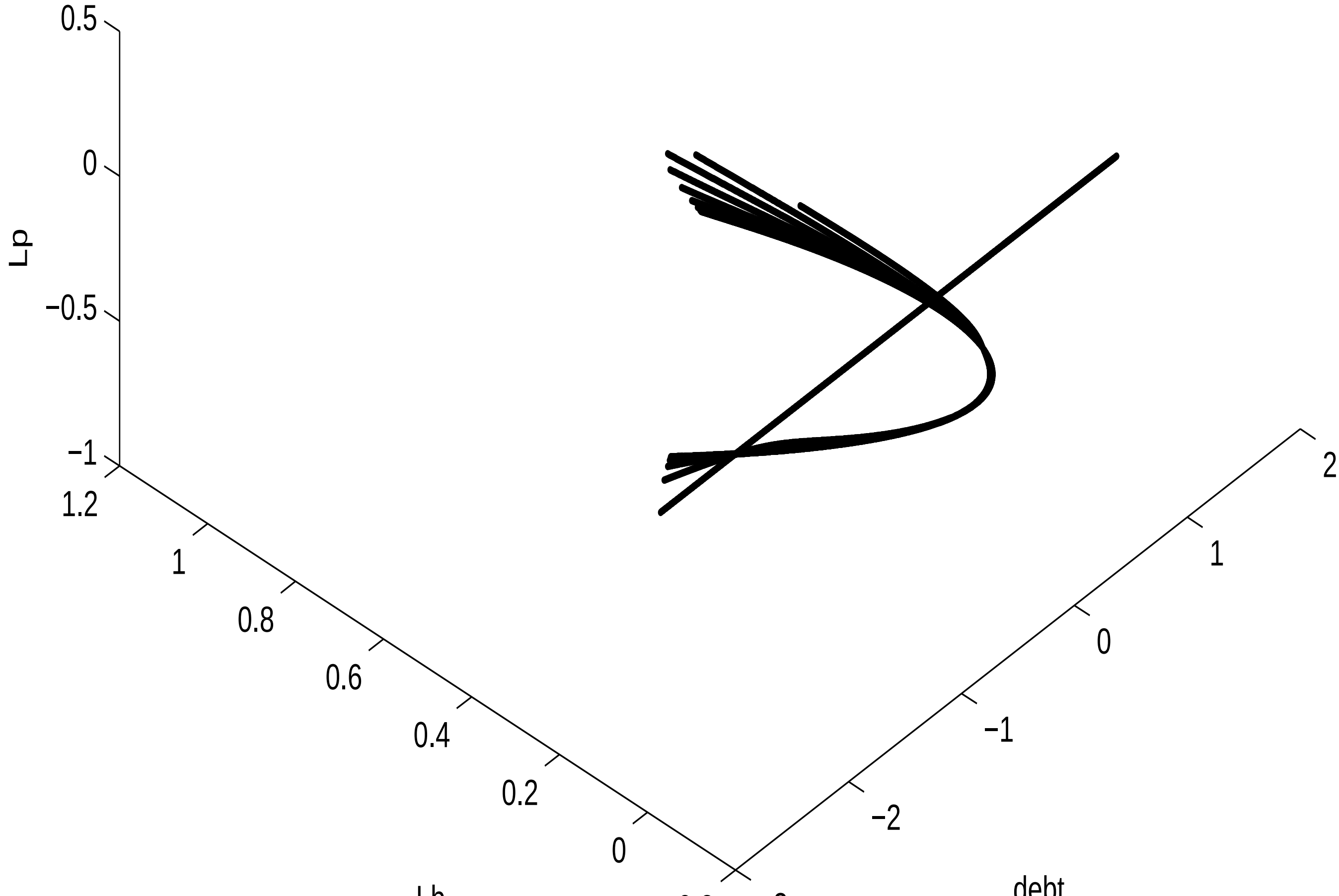

Figure 2: Deterministic model, possible combinations of $(a, \lambda^b, \lambda^p)$

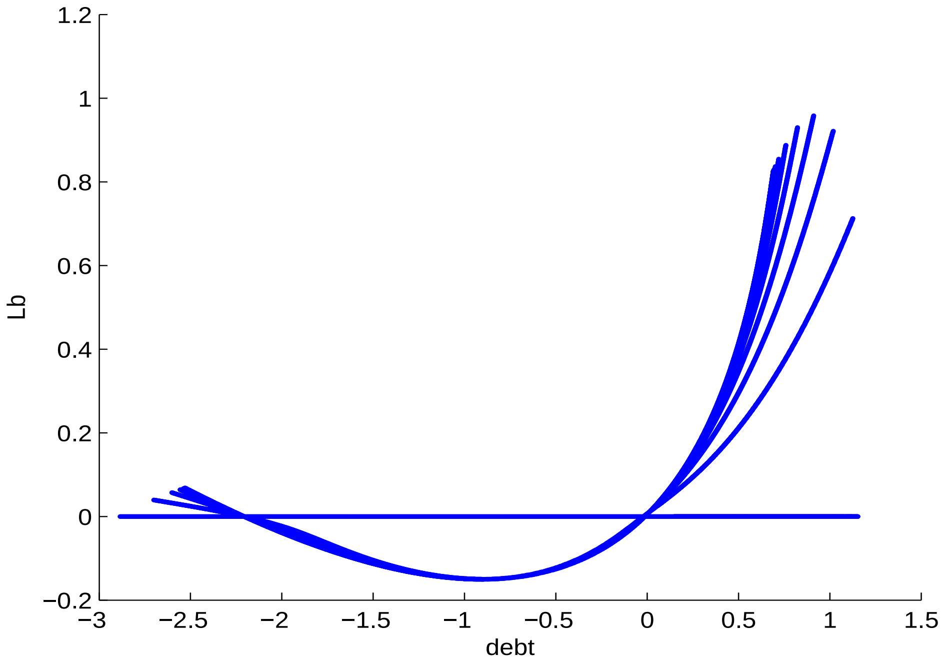

Figure 3: Deterministic model, possible combinations of (a, λ^b)

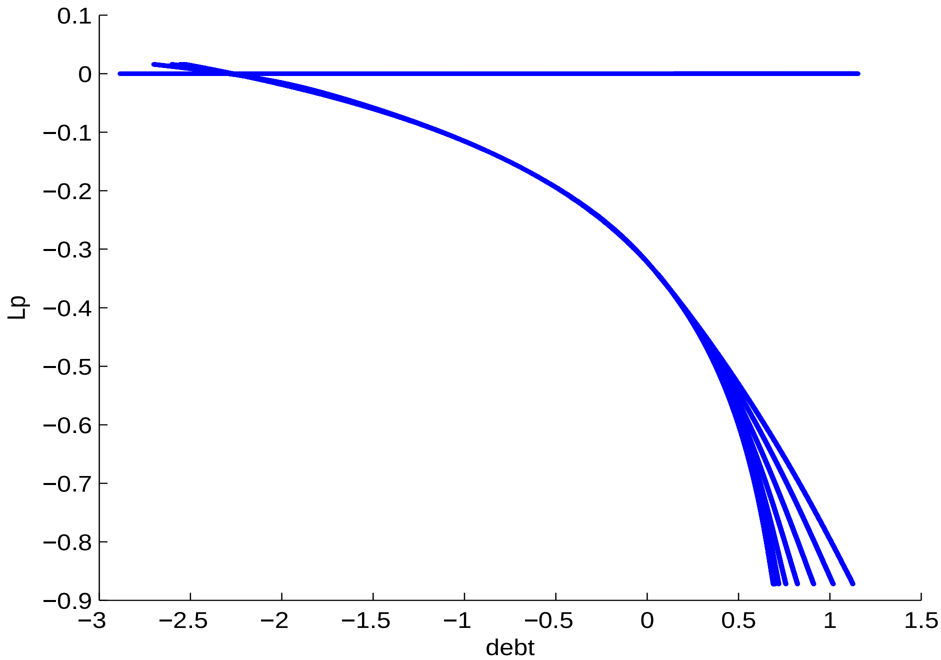

Figure 4: Deterministic model, possible combinations of (a, λ^p)

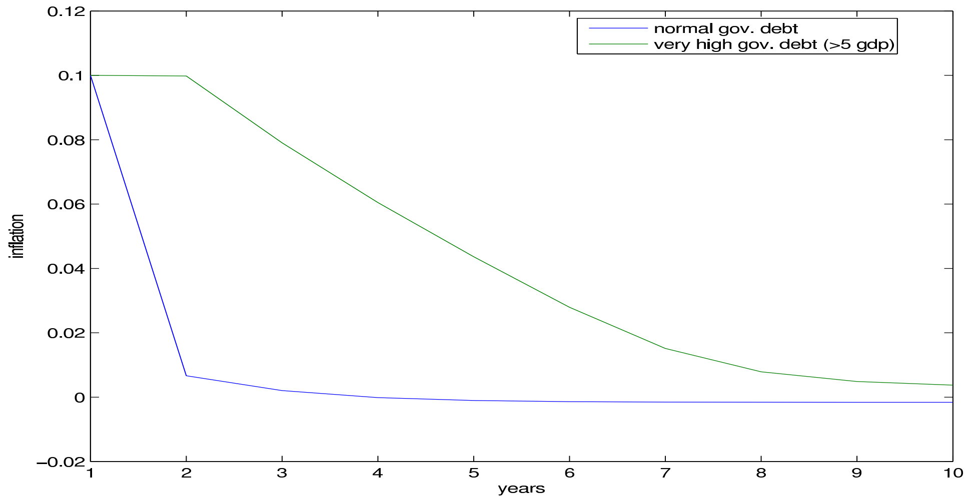

Figure 5: Deterministic model, transition period

For the implementation of the projection method, we have to discuss the following issues:

1. State space on which to approximate
2. Which variables to approximate
3. Basis functions
4. Weighting functions
5. Root-finding algorithm
6. Starting values

4.1.1 State space

A straightforward application that tries to approximate the relevant functions in the natural state space runs into severe trouble. The problem is that a rectangular domain in the state variables $a, \lambda^b, \lambda^p, z, g$, if it is big enough to cover a relevant range of a, contains many extreme, and irrelevant, combinations of a, λ^b and λ^p. The lesson from Figures 2–3 was that, in the deterministic model, only very special combinations of those variable will ever occur, even if we start from a very wide range of possible initial values. With exogenous shocks, we expect to stay rather close to the $(a, \lambda^b, \lambda^p)$-values that appear in the deterministic model, given that aggregate shocks are not huge.

The idea is therefore to approximate the solution on a set that is the Cartesian product of

1. a range of $(a, \lambda^b, \lambda^p)$-values around the set of steady state realizations of $(a, \lambda^b, \lambda^p)$.
2. A reasonable range for the exogenous variables (z, g).

This can be achieved by use of the following **variable transformation**:

For any variable x other than a, define $x^*(a^*)$ as the x in the steady state were $a = a^*$. Then define

$$\tilde{x}_t(a_t) \equiv x_t - x^*(a_t) \tag{13}$$

The state vector is then $S_t \equiv \left(a_{t-1}, \tilde{\lambda}^b_{t-1}, \tilde{\lambda}^p_{t-1}, z_t, g_t\right)$, and the variables we approximate (cf. Section 4.1.2) are also defined in deviations from the respective steady state.

To implement this, I do the following. I compute the deterministic steady states, for a grid of possible values of real government obligations a_i, $i = 1, \ldots, N^{stst}$. I compute the set of relevant variables (states and controls) $X(a_i)$ at any point in the grid. Then I approximate any variable $x \in X$ as a function of a by fitting a cubic spline through the values $x(a_i)$ on the grid. The spline approximation then serves for $x^*(a)$ in (13).

4.1.2 Variables to approximate and residuals

At each point in the state space $S \in \mathcal{S}$, five variables have to be approximated: $c(S)$, $mc(S)$, $\Pi(S)$, $\nu(S)$ and $\lambda^s(S)$. Each approximation is a linear combination of the basis functions laid out in Section 4.1.3. To those 5 variables, there are 5 first order conditions (the derivative of the Lagrangian w.r.t. a_t, λ^b_t, λ^p_t, mc_t and ν_t). These are the residuals that are then minimized by the projection method.

Given S and those five variables, the value of next period's endogenous state variables a_t, λ^b_t and λ^p_t, can be solved linearly from the three remaining first order conditions (the derivative of (11) w.r.t. c_t, Π_t and λ^s_t).

4.1.3 Basis functions

In our state vector $S_t \equiv \left(a_{t-1}, \tilde{\lambda}^b_{t-1}, \tilde{\lambda}^p_{t-1}, z_t, g_t\right)$, a is a variable that can wander around a lot (roughly a random walk), while the other variables should be stationary (this is the advantage of $\tilde{\lambda}^b_t, \tilde{\lambda}^p_t$ over λ^b_t, λ^p_t). Then it is a natural idea to use as basis functions the tensor product of

- quadratic or higher complete polynomial basis in $\tilde{\lambda}^b_t(a_t), \tilde{\lambda}^p_t(a_t), z_t, g_t$ about the point $(0, 0, z^*, g^*)$.
- higher order polynomial or spline basis in a

4.1.4 Minimum residuals

The residuals defined in Section 4.1.2 are minimized in the sense of the Galerkin method (Judd 1992). To compute the integrals I have tried both quadrature grids and the interpolation grids, but the results are very close.

4.1.5 Root finding

I use an implementation of Broyden's algorithm similar to Press, Flannery, Teukolsky and Vetterling (1986, Section 9.7), somewhat adapted and translated into Matlab syntax.

4.1.6 Starting values

For practical purposes, it is extremely important to get good starting values for the approximation parameters.

1. A (log-)linear approximation about any of the deterministic steady states can be obtained in the usual way (I used Uhlig's toolkit, any other toolkit would do it).

2. For quadratic or higher approximations about any deterministic steady states, we obtain starting values from the linear approximation in the following way: Choose a grid of points G in the relevant neighbourhood of the steady state. For the approximation of any variable c, compute the values of the linear approximation of c at the points of G. Collect the values in the vector y. Compute the values of all the basis functions at the points G, and collect them in the matrix B. Compute the coefficients of the approximating function by the linear regression $(B'B)^{-1}B'y$.
3. To get starting values for the global approximations, I did the following steps:
   - I choose a grid of points a_j, $j = 1, \dots, N$, at each of which I compute a quadratic approximation of the control variables (second degree complete polynomial) of the solution, as explained above.
   - I choose a grid of points in $\tilde{\lambda}_t^b, \tilde{\lambda}_t^p, z_t, g_t$, denoted by $s_1, \dots, s_n$. Then, for each $j = 1, \dots, N$, each $l = 1, \dots, n$ and each relevant variable x, I select an estimate $x^*(a_j, s_l)$ as the value of x obtained from the local quadratic solution at the steady state of a_j.
   - I compute an estimate $\hat{x}(a, s)$ by fitting a flexible functional form to the points $x^*(a_j, s_l)$. Concretely, as a basis for the function approximation, I was using the tensor product of a B-spline basis in a and a complete second degree polynomial basis in s. Other choices are possible, of course.
   - For a suitable grid in the state space of the global approximation, I use the $\hat{x}(a, s)$ to approximate the desired functions, and then use least squares as explained above to obtain the coefficients of the approximating polynomials.

4.2 Results

In the following, I show results for the following approximations:

- Results from loglinearization about the steady state with debt (excluding money) equal to 0.88, which is 44% of GDP. Denoted "Linear" in the graphs.
- Results of a quadratic approximation (complete polynomials of degree 2) in a neighbourhood of the same steady state. The variances of the exogenous shocks are set to their calibrated values, and the state space for the exogenous variables chosen accordingly. Denoted "Quadr. with uncert." in the graphs.
- Results from an approximation in a neighbourhood of the steady state curve, along the steady states where total debt (including money) ranges from 0 to 0.4 (roughly, from 0 to 200% of GDP). The approximating functions are the tensor product of polynomials of order 6 (degree 5) in a, and complete polynomials of order 3 to 5 (quadratic, cubic and degree 4). In the graphs, they are denoted "Global, order 6/3" etc.

To show that the higher order terms really lead to an increase in accuracy, Figure 6 shows the Euler residuals that arise from those different solution. More precisely, I simulated the model many times for 100 years, and at each point that we reach in the simulation, I compute the Euler residual (the residual of each of the 5 first order conditions that we use in the projection method; note that we have eliminated 3 of the first order conditions, since they serve to solve linearly for 3 of the variables). Figure 6 shows the average of the absolute residuals, as a function of the time in the simulation. Conclusions:

- The quality of the linear and quadratic approximations decreases in the course of the simulation. This was to be expected, since we approximate about one steady state, and over time the economy wanders away from this steady state.

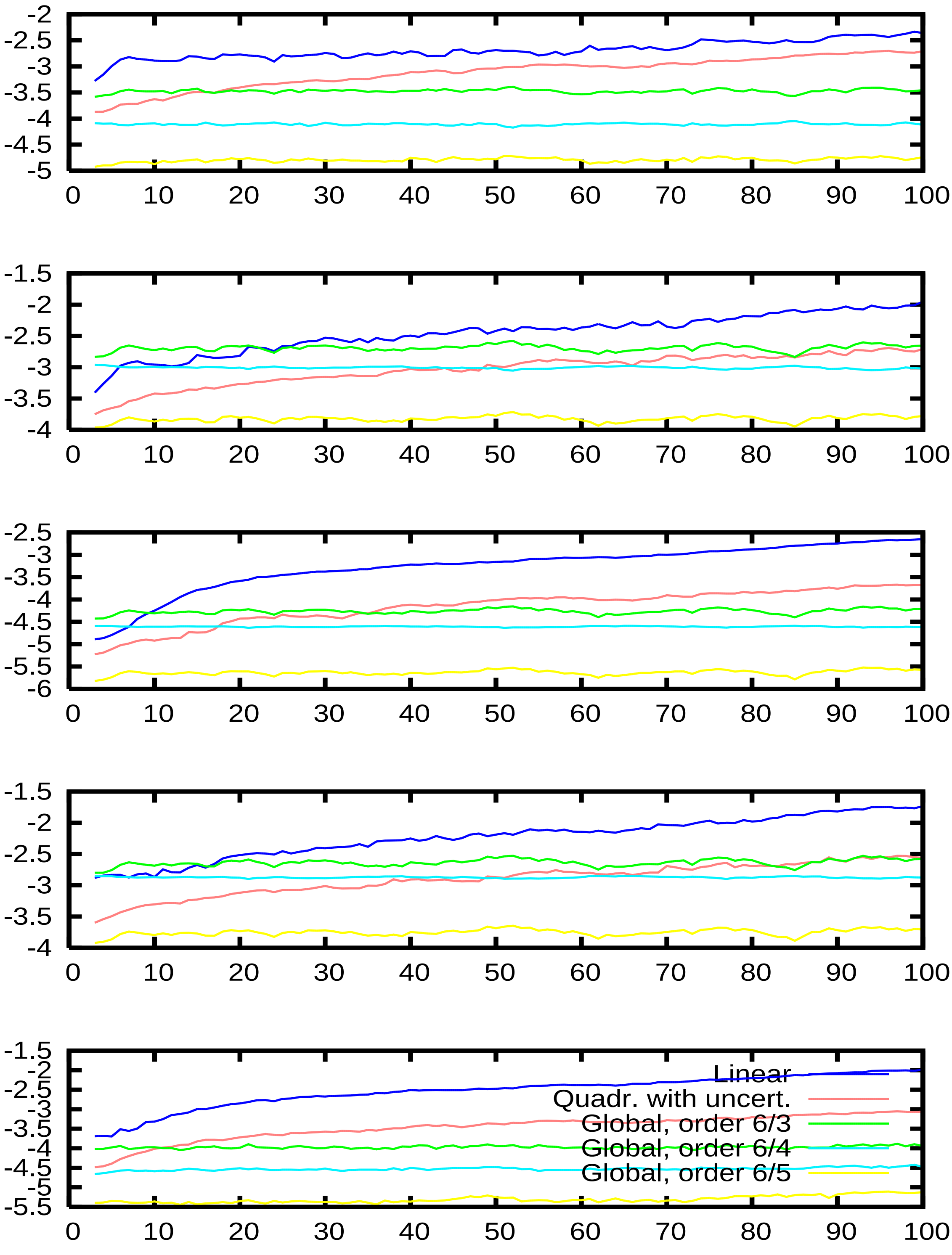

Figure 6: log10 of average absolute Euler residuals, years of simulation

- The accuracy of the "global" approximations stays about constant over time.
- Higher order brings higher accuracy. Solution "6/5" has residuals that are 1 or 2 orders of magnitude smaller than those of the locally quadratic approximation (and 2 or 3 orders of magnitude better than the loglinear one).

The next question is whether accuracy matters for the results. Table 1 gives simulation results of the model "Global, order 6/4". It should be compared to "Baseline sticky

Variable	Mean	Std.dev.	Autocorr.	Corr. with GDP
τ	0.2515	1.0264	0.7474	-0.2557
Infl.	-0.1581	0.1761	0.0328	-0.0894
Intr.rate	3.8360	0.5609	0.8613	-0.9457
y	0.2085	0.0072	0.8177	1.0000
h	0.2084	0.0025	0.8144	-0.0837
c	0.1682	0.0070	0.8206	0.9360

Table 1: Simulation results, approximation "Global 6/4"

price economy" part of Table 2 of Schmitt-Grohé and Uribe (2004a). We see that the results are basically identical to those in the paper. This means, the additional accuracy plays no role for the business cycle statistics are considered here.

Note the exact interpretation of the numbers in the table: they are sample moments (means, standard deviations etc.) of simulations over 100 years (not standard deviations of changes, but of absolute levels), without applying any filter. Although the variation of tax rates (1 percentage point) is big compared to the variability of inflation, it is still very small compared to the data: over a horizon of a century, we observe tax changes that are much bigger than what we expect from a 1 percentage point standard deviation. Since taxes are nonstationary, the standard deviation as reported in the table increases in the time horizon (for 200 years, it would be about 1.4 percentage points). This is not true for R and Π, which are approximately stationary, cf. Figure 1. The small variations in variables help explain why even the simplest approximations are reasonably precise in this model, with this calibration. The overall fluctuations are relatively small; if the economy were moving further around in the state space, the inaccuracies would add up much faster. Ongoing research shows that parameterizations of the model that yield more realistic long-run fluctuations of the model might change those conclusions.

Let us now look at a "typical" simulation of the model, as shown in Figure 7 (in fact, the simulation is not really "typical", debt moves away from the steady state here more than it does on average). We see that in the first years of the simulation, all solutions gives almost identical results, while after about 50 years, the local approximations drift slowly away. But it is no surprise that the business cycle statistics shown in Table 1 are not affected by this.

Let us finally look at the question of government precautionary saving. Figure 8 shows a simulation where all shocks are zero. We see the precautionary savings because the government systematically reduces government debt in this case. This holds true for all approximations except (of course) the linear one, which gives a certainty-equivalence solution where precautionary motives don't enter. However, the results are somewhat disappointing in two respects. First, the precautionary saving is very small, we are talking about a reduction of debt of about 0.1% of GDP per year. Second, this small number is difficult to pin down: increasing the approximation order, the result seems not yet to converge. It seems that for this subtle question even higher accuracy is needed.

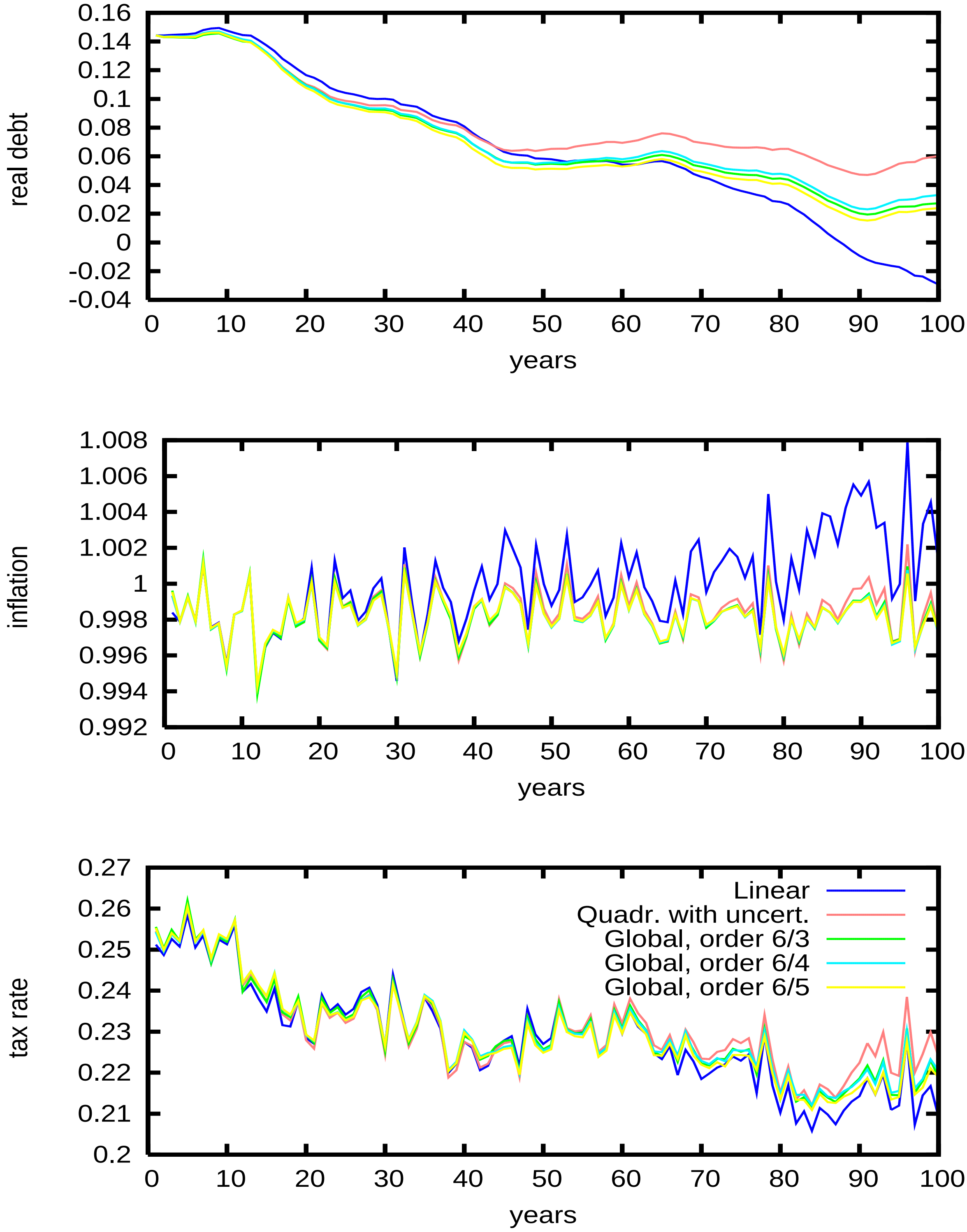

Figure 7: Typical simulation of some variables

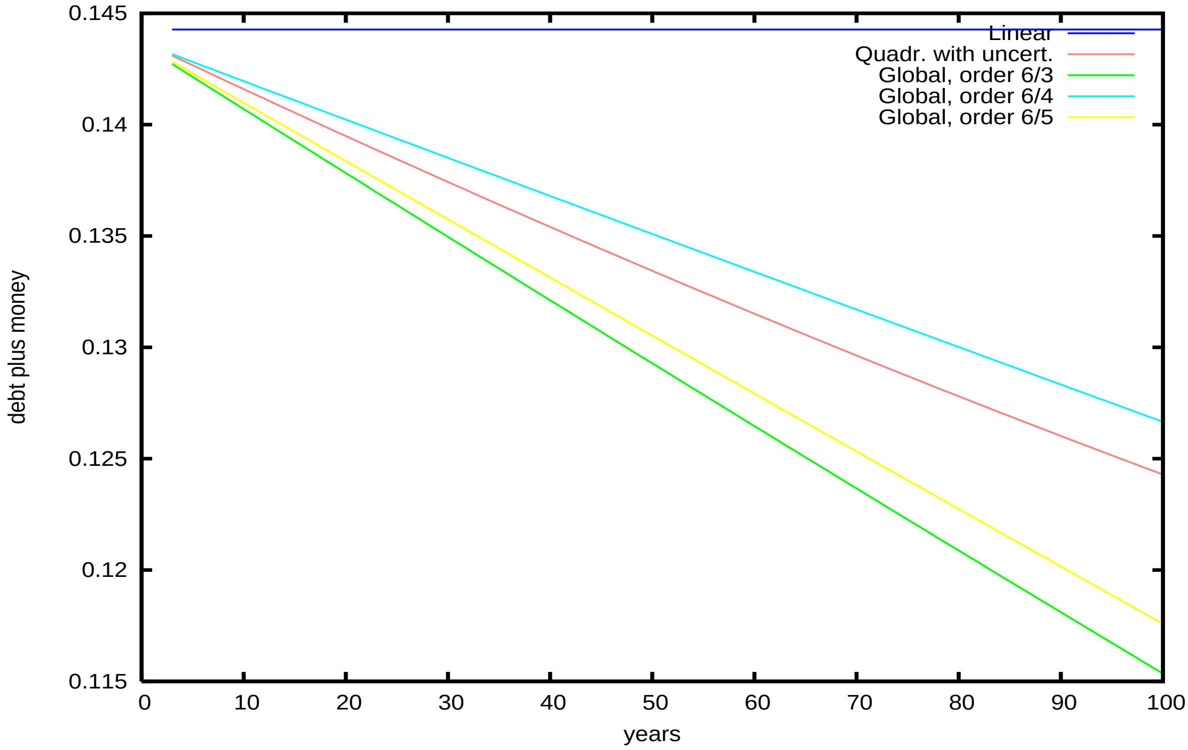

Figure 8: Simulation with zero shocks

4.3 Value functions

We have seen that in the current model, with the current parameters, even the simplest approximations like the linear and quadratic approximations about some steady state, are reasonably precise. A next step is to see whether this holds true also for the value functions that are implied by different approximations.

For that, we solve the Bellman equation

$$V(S;\Psi) = U(c(S;\Psi), h(S;\Psi)) + \beta \int V(S';\Psi)\, Q(S, dS';\Psi) \tag{14}$$

The parameter Ψ indicates that the value function V is based on the approximation Ψ, which affects V both by determining current consumption and labor, and the transition law from state S to next period's state S' by the transition law Q. To facilitate the comparison of V for different approximations Ψ, I always choose the same approximation type for V, namely a polynomial of degree 4 in S.

Figure 9 shows results for 4 different approximations: linear approximation about the steady state, quadratic approximation (accounting for uncertainty) about the steady state, a global approximation of order 6/3 and a global approximation of order 6/4. In each subplot, one state variable is varied over the range of the approximations, while all other state variables take on their value in the benchmark steady state. Note that the Lagrange multipliers are expressed as deviations from steady state, that means, we use the transformations $\tilde{\lambda}^b_t, \tilde{\lambda}^p_t$.

The value function was normalized such that changes in value can be interpreted as permanent proportional changes in consumption. We see that starting from a government debt (plus money) of 0.2, roughly equal to 100% of annual GDP, brings a value loss equivalent to about a one percent permanent decrease in consumption, compared to a starting situation where debt equals 0.1.

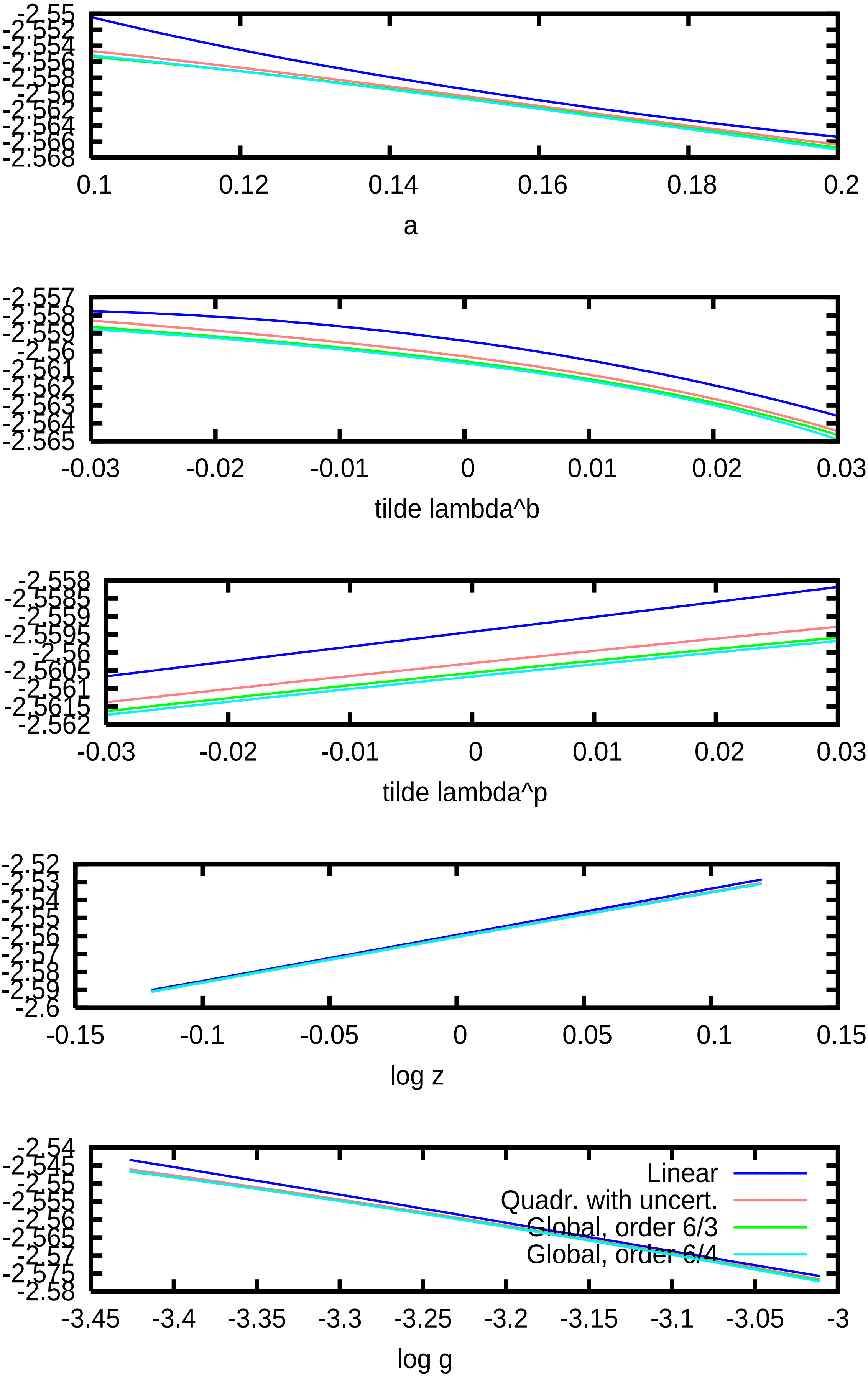

Figure 9: Value functions

The figure shows that all approximations but the linear one give very similar results. And even the linear one differs in most cases only by approximately a constant. The only case with a more substantial deviation is a variation in debt over a relatively wide range, where the linear approximation indicates a wrong shape of the value function.

5 Solving for the transition path

So far we have solved for the full Ramsey solution of the deterministic model (solution starting in period 1), and for an approximation of the stochastic model close to the range of steady state values.

We might finally want to find the full Ramsey solution of the stochastic model. This can be done by approximating the solution around a given deterministic transition path, starting at $s_0 = (a_0, 0, 0)$ (we denote the vector of state variables in the deterministic model by s_t, and those of the full model by S_t; note that the Lagrange multipliers are zero in s_0). As explained in Section 3.2, we compute the deterministic solution path $s_0, s_1, \ldots, s_T$, starting from s_0 up to time T such that s_T is in the range of our approximations about the steady state curve. We can then compute an approximation of the stochastic solution of the Ramsey problem, starting at $S_0 = (s_0, z_0, g_0)$ in the following way:

1. Start with the policy functions $P_T(T)$ that come from the steady state approximation.
2. For $t = T-1, T-2, \ldots, 1$, choose a state space as the Cartesian product of a neighbourhood of s_t and $\mathcal{Z}$ (the state space of the exogenous variables).
3. Choose an interpolation grid on this state space. For each point on the grid, compute the policy functions $P_t(t)$ (the variables a_t, λ_t^b, λ_t^p, mc_t, ν_t, c_t, Π_t and λ_t^s) by solving the 8 first order conditions (the derivative of the Lagrangian w.r.t. those variables). If necessary, an inequality constraint on Π must be enforced in the way explained in Section 3.2.

   In the first order conditions, next period's policy functions are given by $P_{t+1}(t+1)$, available from the earlier step in the iteration.
4. Having computed $P_t(t)$ on the grid points, approximate them between grid points by polynomials etc.
5. Goto step 2.

If T is large, which means that we need many steps to reach the steady state, the numerical stability of this procedure may be problematic. In the present model, however, the procedure is very simple, because in the period after the staring period we are already close to a steady state, so that $T = 1$ and we need only one iteration. The implementation of the above procedure then poses no difficulties.

6 Conclusions

The paper has shown that for interesting models of optimal policy, solutions can be computed that are substantially more precise than the first and second order approximations that are usually computed in the literature. It turns that for the model under consideration, and with the calibration used here (as taken from the original source), the simpler methods are sufficiently accurate to answer most of the questions that one wants to address. Ongoing research shows that for other parameterizations of the model (in particular tax rates that reflect the marginal tax rates in Europe), this is not longer clear. This will be documented in future drafts of this paper.

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