Dynamic Screening: A Role for Up-or-Out Contracts*

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Abstract

This paper provides a rationale for the use of up-or-out contracts as a mechanism to induce workers to invest in generating nonverifiable information about their ability. The model consists of a learning game in which a firm and a worker can commonly observe noisy performance signals about the worker's ability. A worker's effort is assumed to improve on the accuracy of the inference process about ability, by increasing the informativeness of the output signals generated. While output is observable but nonverifiable, effort can be either observable but nonverifiable, or unobservable. In particular, retention decisions based on performance cannot be part of a formal contract. We model an up-or-out contract as a commitment, on the part of the firm, to a sequence of retention decisions and to a wage schedule conditional on employment. We identify conditions under which an up-or-out contract is offered in equilibrium.

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1 Introduction

In certain professional service industries, where human capital is a critical input to production, high-performing employees are awarded virtual permanent tenure, after having spent some time working at a firm at a lower-paying job position. The use of tenure systems coupled with up-or-out clauses is widespread in law, medicine, advertising and accounting partnerships, in academic departments and hospitals. Formally, up-or-out contracts are arrangements between a firm and a worker with the following features: (*i*) the firm commits to retain the worker for a pre-specified period; (*ii*) the worker is considered for promotion only at the end of the probationary period. If promoted, he is granted permanent retention, usually at a higher and better-paying position.[1] If not promoted, he is permanently fired.

In an incomplete information framework, the commitment to a period of probationary employment, and the award of permanent retention upon promotion, seems puzzling. Whenever a firm does not observe a worker's relevant productive characteristic, e.g., his ability, and uncertainty about it persists over time, the possibility of firing the worker, in case his performance proves repeatedly unsatisfactory, is intuitively beneficial. In these instances, when deciding whether to employ a worker, the firm trades-off the benefit of improving on its assessment of the worker's ability against the cost of employing a worker whose talent might be inadequate. As long as the firm efficiently weighs the value of new information about ability against the risk of short-run losses, separation after bad performance is profitable. An up-or-out contract, therefore, entails the intrinsic risk of a loss for the firm, given that the firm specifically commits to forgo the possibility of firing a worker found incompetent.

In this paper, we investigate an economic rationale for the optimality of up-or-out contracts, which motivates their use in firms in which technology is most sensitive to a worker's ability. The interpretation that we propose is that, by committing to such a contract, a firm can induce workers to invest in generating nonverifiable information about their unobserved ability. This information, in turn, enhances the firm's screening power. Specifically, the commitment to employ workers for a pre-specified number of periods, with an implicit promise of permanent retention only to the ones who perform best, stimulates workers to exert effort and produce output realizations which are on average more informative about their true productivity.

The model consists of a dynamic game between an infinitely-lived firm and a pool of finitely-lived workers. Workers can be of two ability levels, unobserved to both the worker and firm. The firm has the ability to commit to a sequence of probationary periods of employment and wage payments. A

[1] Commonly, the employer can dismiss the worker only in extreme cases of misbehavior on the part of the worker, for instance, severe moral misconduct or malpractice.

worker, on the other hand, has always the possibility of quitting the firm and collecting an outside option. An employment contract is modelled as a sequence of retention periods, with the firm paying a worker, upon retention, no less than a pre-specified contractual wage. At the end of each retention period, which can consist of more than one calendar dates, the worker is considered for employment for the next period of retention. Output is an imperfect signal of ability, and it is observable but non-verifiable: retention decisions based on performance cannot be part of a formal contract. Effort, on the other hand, can be either observable but not verifiable, or unobservable.

Crucially, only when the worker exerts effort on the job, he can produce a certain output signal. This output realization is more likely to occur when the worker's intrinsic ability is high rather than low. Moreover, it is a more accurate signal about ability that the ones generated when the worker chooses no effort. Given this informational structure, we provide conditions under which an up-or-contract with one period of probation is offered in equilibrium. Even workers who are permanently retained exert effort only during the probationary period. In particular, the contract is shown to improve on the best sequence of one-period employment contracts. This random sequence of spot employment contracts is also the (essentially unique) equilibrium outcome of the game in which the firm can commit to employment only for one period. In this case the firm's uniquely optimal employment strategy is characterized by a sequence of increasing reservation beliefs and prescribes that the firm offer employment in a period only if the worker's assessed ability is sufficiently high.

The optimality of up-or-out contracts derives from the firm's willingness to risk retaining a low ability worker for the benefit of inducing a more accurate screening early in a worker's career. Specifically, the benefit from the contract is associated with the expected informational gain from effort exertion, which, in case the contract is offered in equilibrium, is proved to offset: (i) the monetary cost of the wage paid to workers who are eventually retained; (ii) the opportunity cost of retaining a low ability worker. Under the contract, permanent retention is granted only if a sufficiently high output signal is produced by the end of the probationary period. Noticeably, these results are shown to hold even if the expected revenue in a period is lower when the worker exerts effort than when he does not, so that effort has only an informational value.

There are a few related papers that provide an explanation for the use of up-or-out contracts. O'Flaherty and Siow [1992] also interpret up-or-out contracts as a screening device, but their framework allows the firm to dismiss a promoted worker. This eliminates the trade-off of interest under the contract between the improved accuracy of the inference process, early in a worker's career, and the risk of permanent retention of a low ability worker. Other papers rationalize up-or-out contracts as a mechanism to mitigate a form of hold-up problem, involving a worker's investment in general or firm-specific human capital. Along this line of analysis, Kahn and Huberman [1988] motivate up-or-out contracts as a solution to a double static moral-hazard problem. By committing to set a wage higher than a worker's opportunity cost, the firm can induce the worker to invest in firm-specific human capital. The firm's incentive to retain the worker permanently is due to the fact that the worker becomes on average more productive for the firm. Waldman [1990] proves that in an environment in which both a worker's actual and potential employer observe a signal about the worker's productivity, up-or-out contracts provide an incentive to accumulate general human capital. In particular, the firm's retention decision induces competitive bidding for the worker's services by potential employers and this forces the firm to increase post-retention wages.[2] Levin and Tadelis [2004], finally, interpret up-or-out contracts as a commitment device to ensure product quality, if quality is only imperfectly observable in the output market. By dismissing workers who are not most talented, even if they might make a positive contribution to the firm's total profit, a firm can commit to ensure a more efficient level of quality, when public monitoring is imperfect.[3]

The paper is organized as follows. Section 2 presents the model. Section 3 discusses the spot contracting game. Section 4 analyzes the game in which the firm is allowed to commit to employment and compensation for more than one period. Section 5 finally concludes and discusses directions of future research.

2 Basic Framework

Here we describe the firm (the principal) and the workers (the agents). We also state some assumptions that are maintained throughout the paper. The firm is risk-neutral and infinitely lived. Its discount factor is $\delta \in (0, 1)$ and it has an outside option that we set to zero.[4] Workers are risk-neutral as well, but, unlike the firm, they live for 3 periods only. We assume that their discount factor is the same as the firm's.[5] They also have an outside option, that we denote by U, where $U \geq 0$. The per-period payoffs of both the firm and the workers are normalized by $(1 - \delta)$.

[2]Carmichael [1988] proposes an environment in which up-or-out contracts, by guaranteing permanent tenure, provide incumbent workers with the appropriate incentive to select the best junior workers. Harris and Weiss [1984] analyze a two-tier job market within a matching framework and show that, being finitely lived, workers who have reached a certain cumulative output record within a given age will remain until retirement in the primary market, in which there is uncertainty about ability. Those who do not, switch at or before that age to the secondary job, where they stay for the remaining of their lifetime.

[3]Along a similar line of analysis, Bar-Isaac [2004] proposes an overlapping generation model of team production and collective reputation, in which promotion to partnership through an up-or-out mechanism provides a young agent with the incentive to exert effort, by rewarding success with the opportunity to take over the firm.

[4]The value of the firm's outside option plays no role in the analysis that follows as long as it is not too high, in which case the firm never hires any worker.

[5]This is not an innocuous assumption. It simplifies the analysis considerably, and presently we don't know how things change when we allow the firm and the workers to have different discount factors.

At the beginning of period 1 there is a finite number of workers of age 1 available to the firm. Moreover, at the beginning of every subsequent period, a new group of age 1 workers becomes available. In any given period, the firm knows the age of all the living workers. Each worker can be of two types, good/high (H) or bad/low (L). A worker's type is not know to both him and the firm, with ϕ_0 being the probability that a worker of age 1 is of the high type.

At every point in time, the firm can employ at most one worker. If employed, a worker chooses whether to exert a costly effort or not. We denote a worker's choice of effort by $e \in \{\underline{e}, \overline{e}\}$, with $\underline{e}$ denoting no effort and $\overline{e}$ denoting effort. The cost of $\overline{e}$ to the worker is $c > 0$. We consider two alternatives in this paper, either effort is observable (by the firm) but not verifiable, or effort is unobservable. It turns out that in both cases the same results are obtained.

When employed, a worker's output per period is stochastic. The possible values to the firm of this output are $y_1 < y_2 < y_3$. If an employed worker chooses $\underline{e}$, the probability of producing y_3 is zero regardless of his type, while the probability of y_2 is $\alpha \in (0, 1)$ if he is of the high type and $0 < \beta < \alpha$ if he is of the low type. If an employed worker of the high type chooses $\overline{e}$, he produces y_3 with probability $1 - \gamma$ and y_2 with probability γ, where $\gamma \in (0, 1)$. If this worker is, instead, of the low type, he produces y_3 with probability $0 < \lambda < 1 - \gamma$, y_2 with probability β, and y_1 with probability $1 - \lambda - \beta > 0$. The following table summarizes the workers' production technology.

Production Matrix

Ability	**Low Effort** ($\underline{e}$)			**High Effort** ($\overline{e}$)		
	y_3	y_2	y_1	y_3	y_2	y_1
H	0	α	$1-\alpha$	$1-\gamma$	γ	0
L	0	β	$1-\beta$	λ	β	$1-\lambda-\beta$

Notice that with the above production technology, if a worker exerts effort and produces y_1, he is immediately revealed as a low type worker. It turns out that this assumption is not crucial for our analysis.[6] For this reason, we don't consider the more general case where the production of y_1 when the worker exerts effort does not reveal the worker's type.

Let $y(\phi, \underline{e})$ be such that

$$y(\phi, \underline{e}) = \phi[\alpha y_2 + (1-\alpha)y_1] + (1-\phi)[\beta y_2 + (1-\beta)y_1] = \phi(\alpha - \beta)(y_2 - y_1) + \underline{y},$$

where $\underline{y} = \beta y_2 + (1-\beta)y_1$. Then $(1-\delta)y(\phi, \underline{e})$ is the expected per-period output of a worker when he exerts no effort and ϕ is the firm's belief that he is of the high type, the firm's belief for short. We assume that $y(\phi_0, \underline{e}) - U > 0$, so that the firm is better off by hiring an untried worker in any

[6]We show this in Section 4, when the analyze the full commitment case.

given period than by collecting her outside option. We also assume that effort is inefficient for both a good and a bad worker; that is, if $y(\phi, \overline{e})$ is the expected output realization of a worker when he chooses $\overline{e}$ and the firm's belief is ϕ, then

$$y(1, \overline{e}) - y(1, \underline{e}) = (1-\gamma)y_3 + \gamma y_2 - \alpha y_2 - (1-\alpha)y_1 < c$$

and

$$y(0, \overline{e}) - y(0, \underline{e}) = \lambda y_3 + \beta y_2 - \beta y_2 - (1-\beta-\lambda)y_1 = \lambda y_3 - (1-\beta-\lambda)y_1 < c.$$

The role of this assumption is to emphasize that effort exertion can be desirable not only because it leads to higher output (a possibility we are ruling out), but also because it leads to output realizations that are more informative about a worker's ability.

3 The Spot Contract Game

In this section we describe and analyze what we call the spot contract game. It turns out that all the equilibria of this game are outcome equivalent; that is, they all imply the same stochastic process over the set of possible firm/worker decisions and output realizations (the outcome space).

3.1 Description

In each period the firm either collects her outside option or offers a worker a wage w in exchange for participation in that period. Participation is verifiable, as well as the wage offer by the firm. The worker then decides whether to participate or not. If he chooses not to participate, he collects his outside option and the firm collects hers. If, on the other hand, the worker chooses to participate, he then makes his effort choice, and output is realized. After this realization, the firm pays the worker his promised wage and chooses whether to pay him a bonus or not. Negative bonus payments are not allowed. Note that when effort is observable, bonus payments contingent on effort exertion and/or output realization are possible, while if effort is unobservable, bonus payments can only be made contingent on output realization. In both cases the firm cannot commit ex-ante to any form of bonus payment.

Workers who are not offered a wage by the firm (and so have to collect their outside option) don't receive any information. In particular, the only way a worker can know if in a given period the firm is engaged in a relationship is if he is the one to receive a wage offer.

3.2 Characterization

Lemma 1. *All equilibria of this game have workers always choosing $\underline{e}$ upon participation, the firm offering $w = U$ when she wants to induce participation, the workers accepting any wage offer greater than or equal to U, and no bonus payments after any possible history (on and off the equilibrium path).*

Proof: By backward induction in the age of an employed worker. First notice that no bonuses are possible for a worker of age 3, as this is his last period of life, and so the firm always reneges on any bonus payment. This is true whether effort is observable or not. Hence, if a worker of age 3 chooses to participate, he chooses no effort, as his future lifetime compensation is independent of his effort choice and output realization in the current period. Consequently, a worker of this age always accepts any wage offer greater than U. This implies that he also accepts any wage offer equal to U, for otherwise no equilibrium would exist. This holds on and off the equilibrium path.

Consider now a worker of age 2 that is employed by the firm. As before, no bonus payments are possible for this worker. To see why, first notice that this worker always accepts participation in his last period of life in case he is offered $w = U$ by the firm. Therefore, if the firm reneges on a bonus payment, this worker will not punish the firm in the following period by not accepting to participate (if the firm indeed wants him to participate). As such, this worker does not exert effort, since once more his lifetime compensation is independent of his effort choice and output realization in the present period. Consequently, a worker of age 2, like a worker of age 3, always accepts any wage offer greater than or equal to U – on and off the equilibrium path.

To finish, consider a worker of age 1. The same reasoning as in the previous paragraph shows that no bonus payments are possible for a worker of this age, and so, for the same reason as before, this worker does not exert effort when employed. Consequently, a worker of age 1 always accepts a wage offer w that is not less than U. □

Lemma 2. *A firm never recalls a worker (still alive) that she disposed of previously.*

Proof: First notice that an untried worker of age k is at least as good to the firm as an untried worker of age $l > k$, with $l, k \in \{1, \ldots, 3\}$. With the younger worker, after $M - l + 1$ periods the firm can choose between this worker and the best alternative available at that point, while with the older worker, the firm is forced to choose the latter.

Denote by W_1 the first worker that is dismissed by the firm and consider the second time in which the firm wants to discontinue her relationship with a worker. Denote by W_2 the worker the firm is employing at this point in time. Without loss of generality, the firm has to choose between an age 1 worker and W_1 (assuming he is still alive). At this point in time, the latter is not better to the firm than he was when dismissed. Moreover, an age 1 worker is, currently, at least as good as an age 1 worker of the previous generations. Hence the firm will replace W_2 with an age 1 worker. An induction argument closes the proof.[7] □

Note that the above two results are valid as long as the workers live for a finite number of periods. The assumption that they live for 3 periods in particular only plays a role in the next section, where we study the full commitment case.

From lemma 2, the only payoff relevant information for the firm at the beginning of every period is either the pair (k, ϕ), where $k \in \{2, 3\}$ is the current age of the last worker she employed and $\phi \in [0, 1]$ is her present belief that this worker is of the high type, or ϕ_0, the probability that an untried worker of any age is good.

From lemma 1 we know that no bonus payments are feasible, an employed worker never exerts effort, and the cheapest way to induce a worker to participate is to offer him $w = U$. Moreover, by assumption, we know that the firm never collects her outside option. In this case, the problem of the firm consists in choosing, in every period, whether to pay $w = U$ to retain the worker she employed in the previous period, if there is such a worker, or offer the same wage to an untried worker. If the firm decides for the second alternative, she also needs to choose the age of the untried worker. At the end of this section we are going to see that the firm never hires workers of ages 2 and 3.

Let $q : [0, 1] \to \mathcal{P}[0, 1]$ be the transition probability that maps the firm's current belief about a worker into the distribution of possible updated beliefs when the firm employs this particular worker and he exerts no effort. The Bellman equations for the firm's problem are then given by

$$V(3, \phi) = \max \Big\{ (1-\delta)[y(\phi, \underline{e}) - U], (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta \int V(2, s) q(ds|\phi_0), \\ (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta V(\phi_0), (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta \int V(3, s) q(ds|\phi_0) \Big\}, \quad (1)$$

$$\begin{aligned} V(2, \phi) \quad = \quad \max \Big\{ & (1-\delta)[y(\phi, \underline{e}) - U] + \delta \int V(3, s) q(ds|\phi), (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta V(\phi_0), \\ & (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta \int V(2, s) q(ds|\phi_0), \\ & (1-\delta)[y(\phi_0, \underline{e}) - U] + \delta \int V(3, s) q(ds|\phi_0) \Big\}, \end{aligned} \quad (2)$$

[7] Observe that it is irrelevant for this argument whether the firm fires a worker when she is indifferent between him and the best available alternative or not.

$$V(\phi_0) = \max\left\{(1-\delta)[y(\phi_0,\underline{e})-U]+\delta V(\phi_0),(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V(2,s)q(ds|\phi_0),\right.$$
$$\left.(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V(3,s)q(ds|\phi_0)\right\}. \quad (3)$$

If the firm employed a worker of age 2 in the previous period – so that in the present period he is of age 3 – and her current belief about this worker is ϕ, her payoff from employing him in the present period is $y(\phi,\underline{e})-U+\delta V(\phi_0)$, since in the next period the firm is forced to look for an untried worker. In case the firm decides not to retain this worker, she can choose among untried workers of age 1, 2, and 3. If she chooses an untried worker of age 3, her payoff is $y(\phi_0,\underline{e})-U+\delta V(\phi_0)$, since in the next period she also has to look for an untried worker. If instead she chooses an untried worker of age $k\in\{2,3\}$, the firm's payoff is

$$(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V(k+1,s)q(s|\phi_0),$$

where $k+1$ is the age of this worker in the next period. The first of the above three equations reflects this choice. The second equation has a similar interpretation, the only difference being that in the present period the age of the worker last employed is 2. In period 1 or when the firm employed a worker of age 3 in the previous period, she is forced to choose among untried workers of age 1, 2, and 3. That is the content of the last of the above equations.

A solution to equations (1)–(3) is a triple (V_1,V_2,V_3), where V_1 is a real number and V_2,V_3 are real-valued functions of ϕ. Let $\mathsf{B}[0,1]$ be the set of bounded measurable functions defined on the interval $[0,1]$ endowed with the sup-norm. The following result is then true.

Lemma 3. *Equations (1)–(3) have, for each $\phi_0\in(0,1)$, a unique solution (V_1,V_2,V_3) in $\mathsf{S}=\mathbb{R}\times\mathsf{B}[0,1]^2$. Moreover, this unique solution is such that both V_2 and V_3 are convex increasing functions of ϕ. Finally, $V_2(\phi)\geq V_3(\phi)\geq V_1$ for all $\phi\in[0,1]$.*

Proof: Let $d:\mathsf{S}\times\mathsf{S}\to\mathbb{R}$ be such that

$$d((V_1,V_2,V_3),(W_1,W_2,W_3))=|V_1-W_1|+\sum_{i=2,3}||V_i-W_i||_{\sup},$$

where $||\cdot||_{\sup}$ is the sup-norm on $\mathsf{B}[0,1]$. Then (S,d) is a complete metric space.[8] Now let $T:\mathsf{S}\to\mathsf{S}$ be such that $T(V_1,V_2,V_3)=(T_1V,T_2V,T_3V)$, where

$$T_1V = \max\left\{(1-\delta)[y(\phi_0,\underline{e})-U]+\delta V_1,(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_2(s)q(ds|\phi_0),\right.$$
$$\left.(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_3(s)q(ds|\phi_0)\right\},$$

[8]In fact, d metrizes the product topology on S, and we know that the product of a finite number of complete metric spaces is itself a complete metric space.

$$
\begin{aligned}
T_2V(\phi) &= \max\Big\{(1-\delta)[y(\phi,\underline{e})-U]+\delta\int V_3(s)q(ds|\phi),(1-\delta)[y(\phi_0,\underline{e})-U]+\delta V_1,\\
&\quad (1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_2(s)q(ds|\phi_0),\\
&\quad (1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_3(s)q(ds|\phi_0)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
T_3V(\phi) &= \max\Big\{(1-\delta)[y(\phi,\underline{e})-U]+\delta V_1,(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_2(s)q(ds|\phi_0),\\
&\quad (1-\delta)[y(\phi_0,\underline{e})-U]+\delta V_1,(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_3(s)q(ds|\phi_0)\Big\}.
\end{aligned}
$$

That T is well-defined follows from the fact that q is a transition probability. A straightforward argument shows that T is a contraction with respect to d. Hence, by the Banach fixed point theorem, we know that T has a unique fixed point in S, that we denote by $(V(\phi_0), V(2,\cdot), V(3,\cdot))$.

Let $\tilde{\mathsf{S}}$ be the subset of S such that if $(V_1,V_2,V_3)\in\tilde{\mathsf{S}}$, V_2 and V_3 are convex and increasing functions of ϕ. Then $\tilde{\mathsf{S}}$ is a non-empty closed subset of S, and so if we establish that T maps this subset into itself, a standard argument shows that the unique fixed point of T in S is an element of $\tilde{\mathsf{S}}$. This, however, is a consequence of the following three facts: (*i*) If $\phi_1\geq\phi_2$, then $q(\phi_1)$ first-order stochastically dominates $q(\phi_2)$. Hence, if f is an increasing function of ϕ, then $\int f(s)q(ds|\phi)$ is an increasing function of ϕ as well; (*ii*) If f is a convex function of ϕ, then $\int f(s)q(ds|\phi)$ is also a convex function of ϕ. This follows from the proof of Lemma 3.1 in Banks and Sundaram [1992]; (*iii*) The maximum of any number of convex and increasing functions of ϕ is a convex and increasing function of ϕ as well.

To finish, let $\overline{\mathsf{S}}$ be the subset of S such that if $(V_1,V_2,V_3)\in\overline{\mathsf{S}}$, then $V_2(\phi)\geq V_3(\phi)\geq V_1$ for all $\phi\in[0,1]$. Then $\overline{\mathsf{S}}$ is a non-empty closed subset of S as well. Moreover, $\overline{\mathsf{S}}\cap\tilde{\mathsf{S}}$ is non-empty. If we show that T maps $\overline{\mathsf{S}}\cap\tilde{\mathsf{S}}$ into itself, we are done.[9] First note, from the definition of T_1, T_2, and T_3, that if $V=(V_1,V_2,V_3)\in\overline{\mathsf{S}}$, then $T_1V_1\leq T_2V_2(\phi), T_3V_3(\phi)$ for all $\phi\in[0,1]$. Suppose that $V=(V_1,V_2,V_3)$ is an element of $\overline{\mathsf{S}}\cap\tilde{\mathsf{S}}$. From the monotonicity of

$$
y(\phi,\underline{e})-U+\delta\int V_3(s)q(ds|\phi)\quad\text{and}\quad y(\phi,\underline{e})-U+\delta V_1
$$

in ϕ and from the fact that $V_2(\phi)\geq V_3(\phi)\geq V_1$ for all $\phi\in[0,1]$, we have that if $\phi\leq\phi_0$, then

$$
T_2V(\phi)=(1-\delta)[y(\phi_0,\underline{e})-U]+\delta\int V_2(s)q(ds|\phi_0)=T_3V(\phi).
$$

[9]Once more from the fact that if T maps a closed non-empty subset of S into itself, then the unique fixed point of T must belong to this particular set.

Suppose now that $\phi > \phi_0$. Since $V_2(\phi) \geq V_3(\phi) \geq V_1$ for all $\phi \in [0,1]$, we have that

$$\begin{aligned} T_2V(\phi) &= \max\left\{(1-\delta)[y(\phi,\underline{e})-U] + \delta\int V_3(s)q(ds|\phi), (1-\delta)[y(\phi_0,\underline{e})-U] + \delta\int V_2(s)q(ds|\phi_0)\right\} \\ &\geq \max\left\{(1-\delta)[y(\phi,\underline{e})-U] + \delta V_3(\phi), (1-\delta)[y(\phi_0,\underline{e})-U] + \delta\int V_2(s)q(ds|\phi_0)\right\} \\ &\geq \max\left\{(1-\delta)[y(\phi,\underline{e})-U] + \delta V_1, (1-\delta)[y(\phi_0,\underline{e})-U] + \delta\int V_2(s)q(ds|\phi_0)\right\} \; = \; T_3V(\phi), \end{aligned}$$

where the first inequality follows from Jensen's inequality. Therefore T indeed maps $\overline{\mathsf{S}} \cap \tilde{\mathsf{S}}$ into itself, as we wanted to prove. □

From the above lemma, we have that the Bellman equations (1) and (2) for the firm's problem can be rewritten as

$$V(3,\phi) = \max\left\{(1-\delta)[y(\phi,\underline{e})-U] + \delta V(\phi_0), (1-\delta)[y(\phi_0,\underline{e})-U] + \delta\int V(2,s)q(ds|\phi_0)\right\}, \quad (4)$$

$$V(2,\phi) = \max\left\{(1-\delta)[y(\phi,\underline{e})-U] + \delta\int V(3,s)q(ds|\phi), (1-\delta)[y(\phi_0,\underline{e})-U] + \delta\int V(2,s)q(ds|\phi_0)\right\}. \quad (5)$$

From now on we assume that the firm retains a worker if she is indifferent between him and the best available alternative. It is then immediate to see, from equations (4) and (5), that there are unique interior cutoff beliefs ϕ_2 and ϕ_3 such that the firm retains a (previously employed) worker of age $k \in \{2,3\}$ if, and only if, her belief ϕ about this worker is such that $\phi \geq \phi_k$.

We now prove that $\phi_0 < \phi_2 < \phi_3$. This implies, in particular, that in period 1 and in any other period where the firm needs to look for a new worker, she will always hire a worker of age 1. The following preliminary result is needed.

Lemma 4. *$V_k(\phi_0) = V(k,\phi_0) > y(\phi_0,\underline{e}) - U$ for $k = 2,3$.*

Proof: Let $T : \mathsf{S} \to \mathsf{S}$, where $\mathsf{S} = \mathbb{R} \times \mathsf{B}[0,1]^2$, be the operator introduced in the proof of lemma 3. For any two elements V and W of S, write $V = (V_1,V_2,V_3) \geq W = (W_1,W_2,W_3)$ if $V_i \geq W_i$ for all $i \in \{1,2,3\}$. It is easy to see that T is monotonic; that is, if $V \geq W$, then $TV \geq TW$. Now let $f \in \mathsf{S}$ be such that

$$f = (f_1,f_2,f_3) = (y(\phi_0,\underline{e}) - U, (1-\delta)[y(\phi,\underline{e})-U] + \delta[y(\phi_0,\underline{e})-U], y(\phi_0,\underline{e}) - U)$$

and define the sequence $\{g^n\}$ in S to be such that $g^0 = f$ and $g^{n+1} = Tg^n$ for $n \in \mathbb{N}$. First notice that $T_1 f = f_1$, while

$$T_2 f = T_3 f = \max\{(1-\delta)[y(\phi,\underline{e})-U] + \delta[y(\phi_0,\underline{e})-U], y(\phi_0,\underline{e}) - U\}$$

so that $T_2 f \geq f_2$ and $T_3 f \geq f_3$. Therefore $g^1 = Tf \geq g^0 = f$. Hence, by the monotonicity of T, we have that $\{g^n\}$ is a monotonic increasing sequence; that is $g^n \leq g^{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, since $g_2^1 = T_2 f$ and $g_3^1 = T_3 f$ are convex and have a kink at $\phi = \phi_0$, $T_1 g^1 > y(\phi_0, \underline{e}) - U$ by Jensen's inequality. Consequently $T_2 g^1(\phi_0), T_3 g^1(\phi_0) > y(\phi_0) - U$, and so

$$g^2(\phi_0) = Tg^1(\phi_0) > (y(\phi_0, \underline{e}) - U, y(\phi_0, \underline{e}) - U, y(\phi_0, \underline{e}) - U).$$

To finish, observe, as a consequence of the Banach fixed point theorem, that $\{g^n\}$ converges pointwise to $(V(\phi_0), V(2, \cdot), V(3, \cdot))$, so that $V(k, \phi_0) \geq \lim g_k^n(\phi_0) \geq g_k^2(\phi_0) > y(\phi_0, \underline{e}) - U$ for $k \in \{2, 3\}$, the desired result. □

Lemma 5. $\phi_0 < \phi_2 < \phi_3$.

Proof: We know, from the definition of ϕ_3, that

$$(1 - \delta)[y(\phi_3, \underline{e}) - U] + \delta V(\phi_0) = V(\phi_0).$$

Since, by the previous lemma, $V(\phi_0) > y(\phi_0, \underline{e}) - U$, it must then be that $y(\phi_3, \underline{e}) > y(\phi_0, \underline{e})$. Hence $\phi_3 > \phi_0$, given that $y(\phi, \underline{e})$ is strictly increasing in ϕ.

Because $V(3, \phi)$ is convex and has a kink at $\phi = \phi_3$ we have, from Jensen's inequality, that

$$\int V(3, s) q(ds|\phi_3) > V(3, \phi_3). \tag{6}$$

Therefore, since $V(3, \phi_3) = V(\phi_0)$,

$$(1 - \delta)[y(\phi_3, \underline{e}) - U] + \delta \int V(3, s) q(ds|\phi_3) > (1 - \delta)[y(\phi_3, \underline{e}) - U] + \delta V(\phi_0) = V(\phi_0)$$

Consequently, since

$$(1 - \delta)[y(\phi_2, \underline{e}) - U] + \delta \int V(3, s) q(ds|\phi_2) = V(\phi_0)$$

by the definition of ϕ_2, it must be that $\phi_2 < \phi_3$, given that the left-hand side of the above equation is (strictly) increasing in ϕ.

We now prove that

$$\int V(2, s) q(ds|\phi_2) > \int V(3, s) q(ds|\phi_2). \tag{7}$$

For this let $\phi^y(\phi_2, e)$ be the firm's updated belief about a worker if her original belief about him is ϕ_2, he his effort choice is e, and he produces $y \in \{y_1, y_2\}$. First notice that if $\phi^{y_2}(\phi_2, \underline{e}) \leq \phi_3$, then

$$\int V(3, s) q(ds|\phi_2) = V(3, \phi_2) = V(\phi_0),$$

given that $\phi_2 < \phi_3$. Since – for the same reasons why (6) is true – we have that

$$\int V(2,s)q(ds|\phi_2) > V(2,\phi_2) = V(\phi_0),$$

the desired result holds in this case. Now assume that $\phi^{y_2}(\phi_2,\underline{e}) > \phi_3$. In this case,

$$\int V(k,s)q(ds|\phi_2) = \Pr(y_2|\phi_2)V(k,\phi^{y_2}(\phi_2,\underline{e})) + \Pr(y_1|\phi_2)V(\phi_0)$$

for $k \in \{2,3\}$, where $\Pr(y_i|\phi)$ is the probability the firm assigns to the outcome y_i when her belief is ϕ and the worker exerts no effort. Now observe that

$$V(3,\phi^{y_2}(\phi_2,\underline{e})) = (1-\delta)[y(\phi^{y_2}(\phi_2,\underline{e}),\underline{e}) - U] + \delta V(\phi_0),$$

while

$$\begin{aligned} V(2,\phi^{y_2}(\phi_2,\underline{e})) &= (1-\delta)[y(\phi^{y_2}(\phi_2,\underline{e}),\underline{e}) - U] + \delta \int V(3,s)q(ds|\phi^{y_2}(\phi_2,\underline{e})) \\ &\geq (1-\delta)[y(\phi^{y_2}(\phi_2,\underline{e}),\underline{e}) - U] + \delta V(3,\phi^{y_2}(\phi_2,\underline{e})), \end{aligned}$$

once more from Jensen's inequality. Since $V(3,\phi) > V(\phi_0)$ for any $\phi > \phi_3$, we have that (7) holds in this case as well.

Since the firm always has at her disposal an untried worker of age 2, it is suboptimal for her to hire or retain a worker of this age when her belief about him is less than ϕ_0. This means that $\phi_2 \geq \phi_0$. Suppose then, by contradiction, that $\phi_2 = \phi_0$. In this case we have, from (7) and the definition of ϕ_2, that

$$\begin{aligned} V(\phi_0) &= (1-\delta)[y(\phi_2,\underline{e}) - U] + \delta \int V(2,s)q(ds|\phi_2) \\ &> (1-\delta)[y(\phi_2,\underline{e}) - U] + \delta \int V(3,s)q(ds|\phi_2) = V(\phi_0) \end{aligned}$$

a contradiction. We can then conclude that $\phi_0 < \phi_2$, as desired. □

As stated previously, the above lemma shows that the firm never hires untried workers of ages 2 and 3. This is quite reasonable given that workers live for a finite number of periods: between two ex-ante identical workers, the firm should always choose the one that lives longer. The above lemma also shows that the firm should use a more stringent retention decision for a worker of age 3 than for a worker of age 2. This is also quite intuitive, since for a worker of age 2 the option value of experimentation is positive, while for the older workers it is zero.

To finish this section, we note that it is straightforward to show that $\phi_2 \geq \phi^{y_2}(\phi_0,\underline{e})$; that is, if a worker produces y_2 in his first period of employment, then he is offered employment in the subsequent period. We thus have the following theorem.

Theorem 1. *Suppose that $\alpha(1-\alpha) \leq \beta(1-\beta)$. Then in all equilibria of the spot contract game the following happens: (i) A worker is only retained at the end of his first period of employment if he produces y_2; (ii) A worker is retained at the end of his second period of employment only if he produces y_2 in that period.*

Proof: This result follows immediately from the previous lemma together with the fact that if $\alpha(1-\alpha) \leq \beta(1-\beta)$, then

$$\phi^{y_1}((\phi^{y_2}(\phi_0), \underline{e}), \underline{e}) = \frac{\alpha(1-\alpha)\phi_0}{\alpha(1-\alpha)\phi_0 + \beta(1-\beta)(1-\phi_0)} \leq \phi_0;$$

that is, if a worker produces y_2 followed by y_1 when not exerting effort, the firm's updated belief about this worker is not greater then ϕ_0. □

4 The Full Commitment Case

In this section we consider what happens when the firm can commit to what we call (long-term) contracts. A long-term contract is a list $C = \{w_i, T_i\}_{i=1}^k$, where $k \leq 3$, $w_i \geq 0$ and $\sum_{i=1}^k T_i = A$, the age of the worker to whom C is offered. The wages w_1 to w_k are what we call the committed wages, and we refer to $j \in \{1, \ldots, k\}$ as the j^{th} probationary period. If the list $\{w_i, T_i\}_{i=j}^k$, with $j \geq 2$, is non-empty, we refer to it as a continuation contract.

When the firm offers a contract $\{w_i, T_i\}_{i=1}^k$ to a worker and he accepts this contract, she is committed to the following: (i) For T_1 periods the firm pays the worker, upon participation, a wage of no less than w_1; (ii) If at the end of the first T_1 periods the continuation contract $\{w_i, T_i\}_{i=2}^k$ is non-empty, the firm must decide whether to retain the worker or not. If she retains the worker, she pays him, upon participation, a wage of at least w_2 for T_2 periods; (iii) If at the end of the first $T_1 + T_2$ periods the continuation contract $\{w_3, T_3\}$ is non-empty, then once more the firm has to decide whether to retain the worker or not. If she decides for retention, she pays him, upon participation, a wage of at least w_3 for the rest of the worker's life.

A **standard up-or-out contract** is a list $\{w_1, T_1 = 1, w_2, T_2 = 2\}$. By definition, such a contract can only be offered to (necessarily) untried workers of age 1. Any worker that accepts such a contract is said to be granted **tenure** if he is retained at the end of the first probationary period. For simplicity, we refer to the first probationary period of a standard up-or-out contract as the probationary period only.

Notice that in the class of contracts described above, the rules governing the retention decision(s) of the firm are not part of the contract. Otherwise, this would be equivalent to allowing for output contingent contracts, and in this case we know that the firm is able to induce effort as long as she wants. Notice as well that we are imposing that any long-term contract the firm happens to offer to a worker must specify the terms of the relationship between the firm and this worker for his entire lifetime. In the next section we prove that this assumption is without loss of generality.

In what follows we assume that no committed wage can be smaller than the worker's outside option; that is, no long-term contract $\{w_i, T_i\}_{i=1}^k$ is possible with $w_i < U$. We refer to this assumption as limited liability. In the next section we consider what happens when we drop this restriction.

4.1 The Game

We now describe in the detail the game between the firm and the workers when the firm offers the long-term contracts described above. We refer to this game as the full commitment game. There are three types of periods t:

1. The period t is such that $t > 1$, the firm employed a worker in $t-1$, and no retention decisions have to be made in this period. This means that the firm must employ in t the same worker employed in $t-1$. In this case, like in the spot contract game, the firm offers a wage w to the worker. Since t is, for some $j \in \{2, 3\}$, the j^{th} probationary period of the worker the firm is currently employing, it must be that $w \geq w_j$. The worker then chooses whether to stay or collect his outside option (forcing the firm to collect her outside option as well). In other words, employment is at will. If the worker chooses to participate, he then chooses whether to exert effort or not. Output is then realized and the firm pays a non-negative bonus to the worker.

2. Either $t = 1$ or t is such that in $t-1$ the firm did not employ a worker. In such periods, the firm decides whether to offer a contract C to a worker or collect her outside option. In the first case, the worker that is offered C decides whether to accept it or not. If he rejects the contract, both the worker and the firm collect their respective outside options. If he accepts, then the timing of moves is as in the previous item. Notice that if the worker accepts the contract, the firm cannot offer him a wage less then w_1, the committed wage corresponding to the 1^{st} probationary period.

3. The period t is one where a retention decision has to be made. If the firm decides to retain the worker she employed in the previous period, then the timing of the moves is as in item 1. If, on the other hand, the firm decides not to retain this worker, then the timing of moves is as in item 2. The firm, however, is not allowed to offer a new contract to a worker she just dismissed.

Similarly to the Spot Contract Game, workers who are not offered a contract by the firm don't receive any information.

4.2 Characterization

We first define what we mean by a (time-invariant) anonymous strategy profile. A strategy profile for the full commitment game is said to be **anonymous** if it satisfies the following three conditions: (i) The workers play symmetric strategies; (ii) Whenever the firm has the chance to offer a contract, she either always offers the same contract $C = \{w_i, T_i\}_{i=1}^k$ or she always collects her outside option; (iii) The retention decisions of the firm are always the same, that is, the contingencies that lead to a worker's retention at the end of his i^{th} probationary period, $i \in \{1, \ldots, k\}$, are the same. In particular, in a anonymous strategy profile, the retention decisions for any worker can only depend on his previous output realizations and, in case effort is observable, on his previous effort choices.

Notice that since $y(\phi_0) - U > 0$ by assumption, the anonymous strategy profile where the firm collects her outside option in every period cannot be an equilibrium. So we can assume, without loss of generality, that in an anonymous strategy profile the firm always offers the same contract when she can do so. Notice also that the restriction that $\sum_i T_i = A$, where A is the age of the worker to whom C is offered, implies that in any such strategy profile, only workers of a certain age are offered contracts. Moreover, since there are only untried workers in the first period, only untried workers of a certain age are offered contracts. The final observation we make about anonymous strategy profiles is that in such profiles, the firm's lifetime payoff at any subgame that begins when the firm has to offer a contract, including the game itself, is the same. This fact plays a central role in what follows.

Theorem 2. *The full commitment game has an equilibrium in anonymous strategies.*

Proof: Suppose workers play symmetric strategies; that is, all workers play the same strategy σ_w. It is possible to write the firm's problem as a stationary dynamic programming problem by: (i) Enlarging the state space to include an index that identifies the type of period the firm is in; (ii) Introducing a constraint correspondence that, as a function of the state, identifies what are the possible actions for the firm. Given this, there exists an optimal Markovian decision plan for the firm that is stationary, i.e., the firm has a Markovian stationary best reply to any symmetric strategy by the workers. See Furukawa [1972]. To finish, observe that, if the firm follows such a strategy, there is a symmetric best reply for the workers. $\square$

The next result we establish – the main result of the paper – is that all equilibria in anonymous strategies of the full commitment game are outcome equivalent and are such that: (i) The firm offers a standard up-or-out contract whenever she has the chance; (*ii*) Any worker that accepts this contract exerts effort in the probationary period; (*iii*) A worker is granted tenure only if he produces y_3 in the probationary period; (*iv*) Once granted tenure, the worker exerts no effort.

The approach we employ to establish this result is, through a sequence of lemmas, to rule out as equilibria all anonymous strategy profiles for which (*i*) to (*iv*) above are not satisfied.[10] We begin by establishing the analogue of Lemma 1 for the full commitment game. Notice that it applies to all equilibria of the full commitment game. It says that no bonus payments are feasible under any circumstances and that if a worker accepts a contract, then, as long as he is retained, he is always paid the minimum wage possible (the one(s) specified by the contract).

Lemma 6. *In any equilibrium of this game it must be that: (i) No bonus payments are made after any history; (ii) The wage paid in any period, on and off the equilibrium path, is equal to the wage prescribed by the prevailing contract for that period (assuming that there is one).*

Proof: The proof of this result follows the same line of reasoning used in the previous section to establish Lemma 1. In particular, it makes no difference for the argument whether effort is observable (but not verifiable) or not. □

We now show that the only contracts that are possible in an anonymous equilibrium are the ones with $T_1 = 1$; that is, the ones that stipulate a first probationary period that is one period long. We begin with the $T_1 = 3$ case. Since in any anonymous strategy profile only untried workers are offered a contract, in what follows all workers are assumed to be untried.

Lemma 7. *There is no anonymous equilibrium of the full commitment game where contracts with* $T_1 = 3$ *are offered.*

Proof: From Lemma 6 we know that if any worker accepts this contract then, as long as he stays in the firm, he exerts no effort (and receives w_1). This happens because this worker's continuation payoff is, in any period he is alive, independent of his effort choice. Therefore, if σ is a strategy profile such that the firm always offers a contract with $T_1 = 3$ when possible, the firm's lifetime payoff from σ is at most $V' = y(\phi_0, \underline{e}) - U$, given that $w_1 \geq U$. Consider now the following deviation for the firm:

A. Always offer $C = \{w_1 = U, T_1 = 1, w_2 = U, T_2 = 1, w_3 = U, T_3 = 1\}$;

B. If ϕ is the firm's belief about a worker at the end of his first probationary period, retain him if, and only if, $\phi \geq \phi_2$;

[10] We (I?) conjecture that under the same parameter restrictions for which the main result holds all equilibria of the full commitment game are outcome equivalent to the equilibrium described above.

C. Suppose a worker was retained at the end of the first probationary period, and let now ϕ be the firm's belief about him at the end of his second probationary period. Retain the worker if, and only if, $\phi \geq \phi_3$.

The above strategy for the firm mimics her behavior in the equilibrium of the spot contract game. Given that a worker that is offered C always accepts it and exerts no effort as long as employed, the firm's lifetime payoff from this deviation is $V(\phi_0)$, the firm's (lifetime) payoff in the equilibrium of the spot contract game. To finish note that $V(\phi_0)$ is bigger than V' by Lemma 4, and so the above deviation is profitable for the firm. □

We now consider the $T_1 = 2$ case. We first establish two preliminary results.

Lemma 8. *There is no anonymous equilibrium of the full commitment game where contracts with $T_1 = 2$ are offered are offered to age 2 workers.*

Proof: Let σ be an anonymous strategy profile where contracts with $T_1 = 2$ are offered to workers of age 2. For the same reasons given in the proof of Lemma 7, the payoff to the firm from σ is at most $V' = y(\phi_0, \underline{e}) - U$, in which case the deviation described in the same proof improves the firm's payoff. □

Lemma 9. *In any anonymous equilibrium where a contract with $T_1 = 2$ is offered to age 1 workers, the firm's retention decision is a cutoff rule. In other words, there exists a cutoff belief ϕ_R such that the firm retains the worker at the end of the first probationary period if, and only if, her belief is not less that ϕ_R. Moreover, $\phi_R > \phi_0$.*

Proof: Let σ be an anonymous strategy profile where a contract C with $T_1 = 2$ is offered to age 1 workers, and let V' denote the firm's lifetime payoff at any subgame beginning when she offers C to a new worker. Suppose that ϕ is, at the end of the first probationary period, the firm's belief about a worker to which C was offered. By Lemma 6, this worker exerts no effort if retained. Hence, he is retained if, and only if,

$$(1 - \delta)[y(\phi, \underline{e}) - w_2] + \delta V' \geq V',$$

where w_2 is the post-retention wage specified by C. Hence, the firm's retention decision is indeed a cutoff rule.

Now observe that in order for σ to be an equilibrium, it must be that $V' \geq V(\phi_0) > y(\phi_0, \underline{e}) - U$, given that the firm can always ensure herself a payoff of $V(\phi_0)$ at any subgame where she offers a contract to a new worker. Since $w_2 \geq U$ by limited liability, we can then conclude that ϕ_R must be bigger than ϕ_0.[11] $\square$

Lemma 10. *There is no anonymous equilibrium of the full commitment game that involves: (i) A contract C with $T_1 = 2$ being offered; (ii) The worker that accepts C exerting no effort in the first period of his first probationary period.*

Proof: Let σ be an anonymous strategy profile for which conditions (i) and (ii) in the above statement are satisfied. By Lemma 8, we only need to consider the case where C is offered to an age 1 worker. Now observe, from Lemma 6, that a worker of age 3 never exerts effort when employed. Hence, in principle, there are two cases to consider: either effort is not exerted when the worker is of age 2 or effort is exerted with positive probability when the worker is of this age. For the fist case, however, the discussion in the previous section shows that the deviation described in the proof of Lemma 7, where the firm mimics her behavior in the equilibrium of the spot contract game, is profitable. Therefore, we only need to consider the second case. We first establish that there are only two sub-cases to be considered:

1. The workers exert effort when they are of age 2 only if they produce y_2 in their first period of employment;

2. The workers always exert effort when they are of age 2.

Since the workers are supposed to exert effort in the second period of their first probationary period, there must be some output realizations that lead to retention and others that lead to dismissal, otherwise there are no incentives to exert effort in that period.[12] Moreover, given our assumptions about γ, β, and λ ($\lambda < 1 - \gamma$ and $\beta + \lambda < 1$), the higher is the output realization under effort, the higher is the firm's updated belief about the worker. Hence, if producing $y \in \{y_1, y_2, y_3\}$ when of age 2 leads to a worker's retention, any output realization higher than y leads to retention as well. This follows from the previous lemma. Consequently, there exists $\underline{y}(y') > y_1$, that depends on the first period's output realization y', such that a worker is retained if, and only if, he produces at least $\underline{y}(y')$ in his second period of employment.

Now observe that $\Pr(y_2|H, \underline{e}) > \Pr(y_2|L, \underline{e})$. Hence, the firm's belief ϕ about a worker is higher at the beginning of this worker's second period of employment if he produces y_2 in his first period of employment. Therefore, once more from Lemma 9, $\underline{y}(y_1) \geq \underline{y}(y_2)$, and so $\Pr\{y \geq y(y_2)|\overline{e}\} \geq \Pr\{y \geq y(y_1)|\overline{e}\}$. We can then conclude that if a worker has an incentive to choose $\overline{e}$ after he

[11] Notice that the result just obtained holds even without limited liability, since no worker that accepts C ever stays in the firm in his last period of file if $w_2 < U$. In this case, therefore, any worker that accepts C exerts no effort during his first probationary period, in which case the firms payoff from this strategy profile is $V' = (1 - \delta)^{-1}[y(\phi_0) - U]$.

[12] Besides, it must be that $w_2 > U$.

produces y_1 in his first period of employment, then he has an incentive to choose $\overline{e}$ after y_2 as well. In other words, sub-cases 1 and 2 listed above are indeed the only ones we need to consider.

Let us first analyze the sub-case 2. Given the limited liability constraint, it must be that $C = \{w_1 = U, T_1 = 2, w_2 > U, T_2 = 1\}$. Consider then the following deviation for the firm:

A. Offer $C =' \{w_1' = U, T_1 = 1, w_2' = U, T_2 = 2, w_3' = w_2, T_3 = 1\}$ in the first period. Denote the worker to whom C is offered by W;

B. Retain W from period 1 to period 2 only if he produces y_2 in the first period. Otherwise, offer the contract given by $\{w_1'' = U, T_1 = 1, w_2'' = w_2, T_2 = 1\}$ to an untried worker of age 2. Denote this worker by W';

C. Suppose W is retained from period 1 to 2. In this case, adopt the same retention decision at the end of period 2 as the one used with the original contract.

D. If W is replaced with W' at the beginning of the second period, retain W' from period 2 to 3 only if he produces $y \geq \underline{y}(y_1)$.

E. Behave, from period 4 on if no dismissal occurs in period 3 and from period 3 on if it does occur, in the same way as prescribed by σ.

We argue that this deviation improves the firm's payoff. To see why, first observe that if y_2 realizes in the first period, then the payoff to the firm is the same as in the original strategy profile. If, on the other hand, y_1 realizes in the first period, the firm replaces an age 2 worker with belief $\phi^{y_1}(\phi_0, \underline{e}) < \phi_0$, W, with an age 2 worker with belief ϕ_0, W'. Since in the original strategy profile, W exerts effort after y_1, W' also exerts effort in period 2, as he is more optimistic about himself, is retained in period 3 under exactly the same circumstances as W is retained, and receives the same wage upon retention. Therefore, if y_1 is realized in the first period, the firm's lifetime payoff from period 2 on is strictly bigger with the deviation, as her per-period payoffs are strictly increasing in her beliefs.

The other alternative (sub-case 1) is dealt with in a similar way. As above, the firm should replace the worker she hires in period 1 if he produces y_1. The only difference is that in this case the firm should offer the "flat" contract $\{w_1'' = U, T_1 = 1, w_2'' = U, T_2 = 1\}$ to an untried worker of age 2 at the beginning of period 2. □

Before we state and prove our next result, let us introduce some notation. Suppose that a worker hired by the firm for two periods in a row chooses $e_i \in \{\underline{e}, \overline{e}\}$ and produces $y_{j_i} \in \{y_1, y_2, y_3\}$ in his i^{th} period of employment, $i = 1, 2$. If ϕ is the firm's initial belief about this worker, we denote her updated belief at the end of the two period of employment by $\phi^{y_{j_1}, y_{j_2}}(\phi, e_1, e_2)$.

Lemma 11. *There exists $\underline{\gamma}$ such that if $\gamma \leq \underline{\gamma}$, then the following two facts are true. First, in any anonymous equilibrium where a contract with $T_1 = 2$ is offered and effort is exerted with positive probability in both periods of the first probationary period, retention only happens if y_3 is produced twice. Second, effort is exerted in the second period of the first probationary period only after y_3 is produced in the first period.*

Proof: Let $\underline{\gamma}$ be the smallest solution to the equation $\gamma(1-\gamma) = \beta\lambda$. If $\gamma \leq \underline{\gamma}$, then $\underline{\gamma}(1-\underline{\gamma}) < \beta\lambda$, and so

$$\phi^{y_3,y_2}(\phi_0, \overline{e}, \overline{e}) = \frac{\gamma(1-\gamma)\phi_0}{\gamma(1-\gamma)\phi_0 + \beta\lambda(1-\phi_0)} \leq \phi_0.$$

Therefore, as a consequence of Lemma 9, retention does not happen if y_3 is produced in the first period of the first probationary period and y_2 is produced in the second. Because $\phi^{y_2,y_3}(\phi_0, \overline{e}, \overline{e}) = \phi^{y_3,y_2}(\phi_0, \overline{e}, \overline{e})$, a worker that produces y_2 in the first period of the first probationary period has no incentives to exert effort in the second period, since he is not retained even if y_3 is produced. To finish, notice that retention must happen if y_3 is produced twice, otherwise there are no incentives for effort exertion. □

Corollary 1. *Suppose that $\gamma \leq \underline{\gamma}$. In this case there is no anonymous equilibrium of the full commitment game that involves: (i) A contract C with $T_1 = 2$ being offered; (ii) The worker that accepts C exerting, with positive probability, effort in both periods of the first probationary period.*

Proof: Consider an anonymous strategy profile such that conditions (i) and (ii) in the statement are satisfied. From the above lemma, we know that the worker that accepts C exerts effort in the second period of his first probationary period only if he produces y_3 in the first one, and he is retained only if he produces y_3 in this period as well. Because of limited liability, C must have $w_1 = U$, which implies that a worker that produces y_1 or y_2 in his first period of employment stays in the firm for one more period (without exerting effort). Consider then the following deviation by the firm:

A. Offer $\{w'_1 = U, T_1 = 1, w'_2 = U, T_2 = 1, w'_3 = w_2, T_3 = 1)$ in period 1. Denote the worker to whom C is offered by W;

B. Retain W from period 1 to period 2 only if he produces y_3. Otherwise, offer $\{w''_1 = U, T_1 = 1, w''_2 = U, T_2 = 1\}$ to an untried worker of age 2 and never retain this worker from period 2 to period 3;

C. If W is retained from period 1 to period 2, retain him from period 2 to period 3 only if he produces y_3 in period 2;

D. Behave, from period 4 on if no dismissal occurs in period 3 and from period 3 on if it does occur, in the same way as specified by the original strategy.

It is straightforward to see that this deviation is profitable for the firm, and so the desired result indeed holds. □

Lemma 12. *There is no anonymous equilibrium of the full commitment game that involves: (i) A contract C with $T_1 = 2$ being offered; (ii) The worker that accepts C exerts effort in the first period and no effort in the second period of his first probationary period.*

Proof: Consider a strategy profile such that conditions (i) and (ii) in the above statement are satisfied. Since $\phi^{y_1}(\phi_0, \underline{e}) = 0$, a worker that exerts effort in his first period of employment and produces y_1 is never retained, no matter what happens in his second period. The following deviation is then profitable for the firm, as it allows her, in period 2, to substitute a worker that produces y_1 in the first period with a better worker:

A. Offer $\{w_1' = U, T_1 = 1, w_2' = U, T_2 = 1, w_3' = w_2, T_3 = 1\}$, where w_2 is the post-retention wage in the contract C, in period 1.

B. Retain the worker from period 1 to 2 only if he produces y_2 or y_3. Otherwise, offer $\{w_1'' = U, T_1 = 1, w_2'' = U, T_2 = 1\}$ in period 2 to an untried worker of age 2 and never retain this worker from period 2 to 3.

C. Return to the original strategy in period 2 if a retention happens in that period and in period 3 if a retention does not happen in period 2.

□

Consequently, the only anonymous equilibria possible for this game involve contracts with $T_1 = 1$ being offered.

Lemma 13. *There is no anonymous equilibrium of the full commitment game where a contract with $T_1 = 1$ is offered to a worker of age 3.*

Proof: Since a worker of age 3 never exerts effort when employed, the firm's payoff from such strategy profile is at most $V' = y(\phi_0) - U$, in which case a profitable deviation for her is possible, see Lemma 7. □

It is also true that there is no anonymous equilibrium of the game under consideration where a contract with $T_1 = 1$ is offered to an age 2 worker. Before we establish this we result, we consider the anonymous strategy profiles where a standard up-or-out contract is offered. A straightforward consequence of Lemma 6 is that if a worker (that has been offered such a contract by the firm) is granted tenure, then he exerts no effort afterwards.

Lemma 14. *There is no anonymous equilibrium of the full commitment game where: (i) A standard up-or-out contract is offered; (ii) The worker that accepts it exerts no effort in the probationary period.*

Proof: Consider an anonymous strategy profile σ where (i) and (ii) in the above statement are satisfied. The discussion in the previous section shows that the deviation described in the proof of Lemma 7 is profitable for the firm. □

The above result tell us that standard up-or-out contracts are only possible in an anonymous equilibrium if any worker that accepts them exerts effort in their first period of employment. This, however, is only possible if the difference between w_2, the tenure wage, and U is big enough to compensate the worker for his effort during the probationary period. Since the firm's retention decision is an equilibrium decision as well, we must check that the tenure wages that induce a worker to exert effort during the probationary period are not so high that the firm finds it too expensive to grant tenure.

Consider an anonymous strategy profile where the firm offers a standard up-or-out contract and the workers exert effort during the probationary period. The same reasoning used in the proof of Lemma 9 shows that the firm's tenure decision must be a cutoff rule, with the cutoff belief not lower than ϕ_0. Since $\phi^y(\phi_0, \overline{e}) < \phi_0$ if $y \in \{y_1, y_2\}$, a worker can then only be granted tenure if he produces y_3 during the probationary period. We refer to such a strategy profile as a standard up-or-out profile. The next two results show that if certain parameter restrictions are satisfied, then there are standard up-or-out profiles that are feasible; i.e., the workers' effort choice in the probationary period and the firm's retention decision are both incentive compatible. The first result deals with the case where effort is unobservable.

Lemma 15. *Suppose that effort is unobservable and $\alpha > \frac{(3+\beta)}{4}$. There exist $0 < (1-\alpha)(y_2-y_1) < \overline{c}$, $\underline{\gamma}' \in (0,1)$, $0 < \underline{\phi_0} < \overline{\phi_0} < 1$, $\underline{\lambda} \in (0,1)$, $\underline{\delta} \in (0,1)$, and $\underline{y_3} > y_2$ such that if $\gamma < \underline{\gamma}'$, $\phi_0 \in (\underline{\phi_0}, \overline{\phi_0})$, $\delta > \underline{\delta}$, and $y_3 \in (y_2, \underline{y_3})$, then there are standard up-or-out profiles that are feasible when $c \in ((1-\alpha)(y_2-y_1), \overline{c})$.*

Proof: Let σ be a standard up-or-out profile. Since effort is unobservable, a worker that deviates during the probationary period and exerts no effort – so that he can only produce y_1 or y_2 – is never retained. Straightforward algebra then shows that the IC constraint for the worker's effort decision during the probationary period is given by

$$\delta(1+\delta)p_T[w_2 - U] \geq c \quad \Leftrightarrow \quad w_2 - U \geq \frac{c}{\delta(1+\delta)p_T}, \tag{8}$$

where $p_T = [\phi_0(1-\gamma) + (1-\phi_0)\lambda]$ is the probability of tenure under σ and w_2 is the tenure wage.

Let V be the firm's lifetime payoff from σ. Straightforward algebra shows that if w_1 is the wage in the probationary period and w_2 is the tenure wage, then

$$V = \frac{y(\phi_0, \overline{e}) - w_1 + \delta(1+\delta)p_T[y(\hat{\phi}, \underline{e}) - w_2]}{1 + \delta(1+\delta)p_T},$$

where it must that

$$w_1 + \delta(1+\delta)p_T w_2 + \delta(1+\delta)(1-p_T)U \geq U + \delta(1+\delta)U - c, \tag{9}$$

in order for the workers to accept the standard up-or-out contract.[13] Therefore, an upper bound for V is

$$\overline{V} = \frac{y(\phi_0, \overline{e}) - U - c + \delta(1+\delta)p_T[y(\hat{\phi}, \underline{e}) - U]}{1 + \delta(1+\delta)p_T} = y(\hat{\phi}, \underline{e}) - U - \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + \delta(1+\delta)p_T},$$

where $\hat{\phi} = \phi^{y_3}(\phi_0, \overline{e})$. Therefore, the firm is willing to grant a worker tenure after he produces y_3 in the first period if

$$\begin{aligned}(1-\delta)(1+\delta)[y(\hat{\phi}, \underline{e}) - w_2] + \delta^2 \overline{V} \geq \overline{V} \quad &\Leftrightarrow \quad y(\hat{\phi}, \underline{e}) - w_2 \geq \overline{V} \\ &\Leftrightarrow \quad y(\hat{\phi}, \underline{e}) - w_2 \geq y(\hat{\phi}, \underline{e}) - U - \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + \delta(1+\delta)p_T}.\end{aligned}$$

Rearranging terms in the last inequality, we can then conclude that the firm's retention decision is IC compatible if

$$w_2 - U \leq \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + \delta(1+\delta)p_T}. \tag{10}$$

Consequently, if

$$\frac{c}{\delta(1+\delta)p_T} < \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + \delta(1+\delta)p_T} \tag{11}$$

there exists a tenure wage w_2 such that both (8) and (10) are satisfied. Observe that the above equation is obviously satisfied if we take c sufficiently small. However, effort exertion is assumed inefficient for both the good and the bad workers, and so we cannot simply take c as small as we want. We must also have

$$(1-\gamma)y_3 + \gamma y_2 - \alpha y_2 - (1-\alpha)y_1 = (1-\gamma)(y_3 - y_2) + (1-\alpha)(y_2 - y_1) < c. \tag{12}$$

[13] Observe that under limited liability, condition (9) is automatically satisfied once the IC constraint (8) is satisfied.

and

$$\lambda y_3 + \beta y_2 - \beta y_2 - (1-\beta-\lambda)y_1 = \lambda y_3 - (1-\beta-\lambda)y_1 < c. \tag{13}$$

The last inequality is obviously satisfied is we take λ small enough. So we only worry about (11) and (12). Taking the limit of both inequalities as $\delta \to 1$, $\lambda \to 0$, $\gamma \to 0$, $y_3 \to y_2$, and $\phi_0 \to 1/2$, we have that c must be such that

$$c < \frac{y(1,\underline{e}) - y(1/2,\overline{e}) + c}{2} \Rightarrow c < y(1,\underline{e}) - y(1/2,\overline{e}) = \frac{2\alpha - 1 - \beta}{2}(y_2 - y_1).$$

and $(1-\alpha)(y_2 - y_1) < c$. Therefore, as long as

$$(1-\alpha) < \frac{2\alpha - 1 - \beta}{2} \Leftrightarrow \alpha > \frac{(3+\beta)}{4}$$

the desired result holds. □

Notice, in particular, that in order for the above result to hold, we need $2\alpha - 1 - \beta \geq 0$. From now on we assume that this is always the case (and we know that this is true if $\alpha > \frac{3+\beta}{4}$).

Lemma 16. *Suppose now that effort is observable, that* $(\alpha - \beta)^2 \leq 2(1-\alpha)(\alpha+\beta)$, *and that* $\alpha > (3+\beta)/4$. *There exist* $0 < (1-\alpha)(y_2 - y_1) < \overline{c}$, $\underline{\gamma}' \in (0,1)$, $0 < \underline{\phi_0} < \overline{\phi_0} < 1$, $\underline{\lambda} \in (0,1)$, $\underline{\delta} \in (0,1)$, *and* $\underline{y_3} > y_2$ *such that if* $\gamma < \underline{\gamma}'$, $\phi_0 \in (\underline{\phi_0}, \overline{\phi_0})$, $\delta > \underline{\delta}$, *and* $y_3 \in (y_2, \underline{y_3})$, *then there are standard up-or-out profiles that are feasible when* $c \in ((1-\alpha)(y_2 - y_1), \overline{c})$.

Proof: We know that a worker that exerts effort during the probationary period is only granted tenure if he produces y_3. The firm, however, now observes the worker's choice of effort, and so if he deviates an produces y_2, the firm's belief increases to $\phi^{y_2}(\phi_0, \underline{e})$. Hence, there are two possible alternatives, either the worker is granted tenure if he deviates and produces y_2 or not, and they lead to different IC constraints for the worker's choice of effort. Both alternatives, however, lead to the same payoff V to the firm, where V is the same as in the previous lemma. Therefore, we can, without loss of generality, restrict attention to the case where w_2 is such that

$$y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - w_2 < V, \tag{14}$$

so that a worker that deviates and produces y_2 is not retained. This ensures that the IC constraint for the worker's choice of effort is the same as in the previous lemma. Moreover, we restrict attention to the case where $w_1 = U$ and w_2 is such that (8) binds, so that $V = \overline{V}$.[14]

[14]After all, we know that if a standard up-or-out profile is to be an equilibrium, these two conditions must hold, otherwise there is a profitable deviation for the firm.

Consequently, if (11), (12), (13), and (14) are satisfied, we know that there are feasible standard up-or-out profiles when effort is observable. From the previous lemma and the assumption that $\alpha > (6 + 2\beta)/8$, we know that (11), (12), and (13) are satisfied for a certain range of costs if we take δ close to 1, γ and λ close to zero, ϕ_0 close to 1/2, and y_3 close to y_2. Let $\underline{V}$ be such that

$$\underline{V} = y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - U - \frac{y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - y(\phi_0, \underline{e}) + c}{1 + \delta(1 + \delta)p_T}.$$

It is straightforward to see that $\underline{V} < V$, and so (14) is satisfied if

$$y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - w_2 \leq \underline{V} \;\;\Leftrightarrow\;\; w_2 - U \geq \frac{y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - y(\phi_0, \underline{e}) + c}{1 + \delta(1 + \delta)p_T}$$

is satisfied. Since from equation (8) we must have $w_2 - U \geq [\delta(1 + \delta)p_T]^{-1}c$, we are done if

$$\frac{c}{\delta(1 + \delta)p_T} \geq \frac{y(\phi^{y_2}(\phi_0, \underline{e}), \underline{e}) - y(\phi_0, \underline{e}) + c}{1 + \delta(1 + \delta)p_T}$$

$$\Leftrightarrow \;\; c \geq \frac{\delta(1 + \delta)[\phi_0(1 - \gamma) + (1 - \phi_0)\lambda]\phi_0(1 - \phi_0)(\alpha - \beta)^2(y_2 - y_1)}{\phi_0\alpha + (1 - \phi_0)\beta}.$$

Taking the appropriate limits in the above inequality, it becomes

$$c \geq \frac{(\alpha - \beta)^2}{2(\alpha + \beta)}(y_2 - y_1),$$

which is satisfied – since by assumption $c > (1 - \alpha)(y_1 - y_2)$ – if $(\alpha - \beta)^2 \leq 2(1 - \alpha)(\alpha + \beta)$. $\square$

Lemma 17. *There is no anonymous equilibrium where a contract with $T_1 = 1$ is offered to an age 2 worker.*

Proof: Consider an anonymous strategy profile where a contract with $T_1 = 1$ is offered to an age 2 worker. By Lemma 6, we know that a worker never exerts effort when he is of age 3. Hence we have two cases to analyze:

1. Effort is not exerted when the hired workers are of age 2. In this case, the results from the previous section show that the deviation described in the proof of Lemma 7 is profitable for the firm.

2. Hired workers exert effort when they are of age 2. As above, a worker should only be granted tenure if he produces y_3 in his first period of employment. Therefore, the firm's payoff in such a strategy profile is

$$V' = \frac{y(\phi_0, \overline{e}) - w_1 + \delta p_T[y(\hat{\phi}, \underline{e}) - w_2]}{1 + \delta p_T},$$

where $\hat{\phi}$ and p_T are the same as in the proof of Lemma 15, w_1 is the wage in the first probationary period, and w_2 is the wage in the second. Now observe that we must have

$$w_1 + \delta p_T w_2 + \delta(1 - p_T)U \geq (1 + \delta)U - c$$

in order for any untried worker of age 2 to accept the contract under consideration. Therefore, an upper bound for V' is

$$\overline{V}' = y(\hat{\phi}, \underline{e}) - U - \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + \delta p_T} < y(\hat{\phi}, \underline{e}) - U - \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + (1 + \delta)\delta p_T},$$

where the right-hand side of the above inequality is the payoff to the firm in a standard up-or-out profile where the probationary wage is U and the tenure wage is chosen so that (8) binds. Hence there is a profitable deviation for the firm. $\square$

The next sequence of results identifies conditions under which we cannot have contracts with $T_1 = T_2 = 1$ being offered in an anonymous equilibrium. From the previous lemma, we know that we can restrict attention to (anonymous) strategy profiles where these types of contracts are offered to workers of age 1. The same reasoning used in the proof of Lemma 9 shows that the retention decisions at the end of the first and the second probationary periods must be cutoff decision rules, with the corresponding cutoff beliefs not smaller than ϕ_0.

Lemma 18. *There exists $\underline{\gamma}'' \in (0, 1)$ such that if $\gamma < \underline{\gamma}''$, then there is no anonymous equilibrium where: (i) A contract $C = \{w_1, T_1 = 1, w_2, T_2 = 1, w_3, T_3 = 1\}$ is offered; (ii) The worker that accepts it exerts effort in the first probationary period; (iii) There is an output realization in the first probationary period that leads to retention and after which the worker exerts effort in the second probationary period.*

Proof: Consider an anonymous strategy profile where (i), (ii), and (iii) in the statement above are satisfied and let w_2 denote the wage that C specifies for the second probationary period. From the above paragraph we know that a worker is retained from the first to the second probationary period only if he produces y_3. Hence, as a consequence of condition (iii), he always exerts effort in the latter period.

Once more, let $\underline{\gamma}$ be the smallest solution to $\gamma(1 - \gamma) = \beta\lambda$. From the proof of Lemma 11 and the above paragraph, we have that if $\gamma < \underline{\gamma}$, then any worker that is retained from the first to the second probationary period is retained one more time only if he also produces y_3 in the second probationary period. Let $p'_T = \hat{\phi}(1 - \gamma) + (1 - \hat{\phi})\lambda$, where $\hat{\phi}$ is as above. Since in this case it must be that

$$w_1 + \delta p_T \left[w_2 + \delta p'_T w_3 + \delta(1 - p'_T)U\right] + \delta(1 + \delta)(1 - p_T)U \geq (1 + \delta + \delta^2)U - (1 + \delta)c,$$

in order for C to be accepted, an upper bound for the firm's payoff in this strategy profile is

$$\overline{V}' = (1-\delta) \cdot \frac{y(\phi_0, \underline{e}) - U - c + \delta p_T \left\{ y(\hat{\phi}, \overline{e}) - U - c + \delta p_T' [y(\hat{\hat{\phi}}, \underline{e}) - U] \right\}}{1 - \delta p_T \left[\delta^2 p_T' + \delta(1-p_T') \right] - \delta(1-p_T)},$$

where $\hat{\hat{\phi}} = \phi^{y_3, y_3}(\phi_0, \overline{e}, \overline{e})$.

Straightforward algebra shows that

$$\overline{V}' < \overline{V}'' = \frac{y(\phi_0, \underline{e}) - U - c + \delta p_T [y(\hat{\phi}, \overline{e}) - U - c] + \delta^2 p_T [y(\hat{\hat{\phi}}, \underline{e}) - U]}{1 + \delta(1+\delta) p_T},$$

where $\overline{V}''$ is obtained by setting $p_T' = 1$ in $\overline{V}'$. Therefore, if we let $\underline{\gamma}'$ be such that

$$\delta[y(\hat{\hat{\phi}}, \underline{e}) - y(\hat{\phi}, \underline{e})] = c + y(\hat{\phi}, \underline{e}) - y(\hat{\phi}, \overline{e}),$$

then $\gamma \in (0, \underline{\gamma}')$ implies that the left-hand side of the above equation is smaller than its right-hand side, and so

$$\overline{V}'' < y(\hat{\phi}, \underline{e}) - U - \frac{y(\hat{\phi}, \underline{e}) - y(\phi_0, \overline{e}) + c}{1 + (1+\delta)\delta p_T}.$$

We can then conclude, from the proof of the previous lemma, that if $\gamma < \underline{\gamma}''$, where $\underline{\gamma}'' = \min\{\underline{\gamma}, \underline{\gamma}'\}$, the desired result holds. □

Lemma 19. *Suppose that $\alpha < 2\beta$. Then, for λ and γ sufficiently close to zero, δ sufficiently close to 1, ϕ_0 sufficiently close to 1/2, and y_3 sufficiently close to y_2, there is no anonymous equilibrium of the full commitment game such that: (i) A contract with $T_1 = T_2 = 1$ is offered to an age 1 worker; (ii) Workers never exert effort in their first period of employment; (iii) If employed when of age 2, workers exert effort.*

Proof: Consider an anonymous strategy profile where conditions (i) to (iii) in the above statement are satisfied. Denote by w_3 the wage for the last probationary period. First notice that since $\phi^{y_1}(\phi_0, \underline{e}) < \phi_0$, a worker is only retained from the first to the second probationary period if he produces y_2.[15] Because

$$\phi^{y_2, y_2}(\phi_0, \underline{e}, \overline{e}) = \frac{\alpha\gamma\phi_0}{\alpha\gamma\phi_0 + \beta^2(1-\phi_0)},$$

we have that $\phi^{y_2, y_2}(\phi_0, \underline{e}, \overline{e}) < \phi_0$ for γ sufficiently small. Hence, a worker that is retained from the first to the second probationary period is retained once more only if he produces y_3 in the latter

[15] Retention must occur after y_2 in the first period of employment, otherwise the firm's payoff from the strategy profile under consideration is $(1-\delta)^{-1}[y(\phi_0) - U]$, in which case a profitable deviation is possible for her.

period. Consequently, the on-the-equilibrium-path IC constraint for effort exertion for workers that are employed when of age 2 is

$$\delta\left[(1-\gamma)\phi^{y_2}(\phi_0,\underline{e})+\lambda(1-\phi^{y_2}(\phi_0,\underline{e}))\right](w_3-U)\geq c. \tag{15}$$

Let V' denote the firm's payoff in the strategy profile under consideration. Since we want to find conditions under which this strategy profile is not an equilibrium of the full commitment game, we can restrict attention to the case where V' is greater than or equal to the payoff V the firm obtains in a standard up-or-out profile when (8) binds. In this case, the firm's retention decision from the second to the third probationary period is not IC compatible if

$$y(\phi^{y_2,y_3}(\phi_0,\underline{e},\overline{e}),\underline{e})-w_3<V,$$

as this ensures that the left-hand side of the above equation is smaller than $(1-\delta)V'$. Since

$$V=y(\hat{\phi},\underline{e})-U-\frac{y(\hat{\phi},\underline{e})-y(\phi_0,\overline{e})+c}{1+(1+\delta)\delta p_T},$$

we can rewrite the above inequality as

$$w_3-U>y(\phi^{y_2,y_3}(\phi_0,\underline{e},\overline{e}),\overline{e})-y(\hat{\phi},\underline{e})+\frac{y(\hat{\phi},\underline{e})-y(\phi_0,\overline{e})+c}{1+(1+\delta)\delta p_T}.$$

We can then conclude that if

$$\begin{aligned}\frac{c}{\delta\left[(1-\gamma)\phi^{y_2}(\phi_0,\underline{e})+\lambda(1-\phi^{y_2}(\phi_0,\underline{e}))\right]} &> y(\phi^{y_2,y_3}(\phi_0,\underline{e},\overline{e}),\overline{e})-y(\hat{\phi},\underline{e})\\ &+ \frac{y(\hat{\phi},\underline{e})-y(\phi_0,\overline{e})+c}{1+(1+\delta)\delta p_T},\end{aligned} \tag{16}$$

then it is too expensive for the firm to induce a worker to exert effort if he is retained from the first to the second probationary period: The wage w_3 required is so high that the firm cannot commit to retain the worker if he produces the desired output.

Taking the limits $\gamma,\lambda\to 0$, $\delta\to 1$, $\phi_0\to 1/2$, and $y_3\to y_2$, equation (16) becomes

$$\frac{c}{\phi^{y_2}(1/2,\underline{e})}>\frac{y(1,\underline{e})-y(1/2,\overline{e})+c}{2}\quad\Leftrightarrow\quad c>\frac{\alpha(2\alpha-1-\beta)(y_2-y_1)}{\alpha+2\beta}.$$

Now remember, from Lemma 15, that we must have

$$c<y(1,\underline{e})-y(1/2,\overline{e})=\frac{2\alpha-1-\beta}{2}(y_2-y_1).$$

Consequently, the desired result holds. □

The intuition for this result is straightforward. If α and β are far apart, producing y_2 in the first probationary period leads to a high updated belief, since a worker's output is quite informative about his ability even if he exerts no effort. In this case, the IC constraint (15) is easy to satisfy, making it cheap for the firm to retain a worker from the second to the third probationary period. Therefore, one should expect the above lemma to hold if a worker's output is revealing *only* when he exerts effort.

Lemma 20. *Suppose once again that $\alpha < 2\beta$ and consider an anonymous strategy profile such that: (i) A contract with $T_1 = T_2 = 1$ is offered to age 1 workers; (ii) Workers only exert effort in their first period of employment. Then, for λ and γ sufficiently close to zero, δ sufficiently close to 1, ϕ_0 sufficiently close to 1/2, and y_3 sufficiently close to y_2, the only feasible strategy profiles of the above type are outcome equivalent to a standard up-or-out profile. In other words, any worker that is retained from the first to the second probationary period is not fired afterwards.*

Proof: Consider an anonymous strategy profile where (i) and (ii) in the above statement are satisfied. We have two cases to rule out:

1. Suppose that the firm never retains a worker from the second to the third probationary period. If effort is unobservable, the IC constraint for effort exertion in the first period is

$$\delta[(1-\gamma)\phi_0 + \lambda(1-\phi_0)](w_2 - U) \geq c. \tag{17}$$

Since $\phi_0 < \phi^{y_2}(\phi_0, \underline{e})$, the above IC constraint is more stringent then the IC constraint for effort exertion (15) of the previous lemma. Hence, if $\lambda, \gamma, \delta, \phi_0$, and y_3 satisfy the conditions in the above statement, the firm's retention decision from the first to the second probationary period is not IC compatible.

Suppose now that effort is observable. We have two alternatives for the off-the-equilibrium-path behavior of the firm. Either she retains a worker that deviates in the first probationary period and produces y_2 or not.[16] The second alternative leads to an IC constraint for effort exertion in the first probationary period identical to (17). So we can, without loss of generality, consider only the first alternative. In this case the IC constraint for effort exertion is

$$\delta[(1-\gamma-\alpha)\phi_0 + (\lambda-\beta)(1-\phi_0)](w_2 - U) \geq c,$$

which is more stringent than (17).

2. Suppose now that a worker is retained from the second to the third probationary period only if the produces y_2 in the second probationary period. Moreover, suppose that effort is unobservable.

[16] If the worker deviates and produces y_1, the firm's updated belief about this worker is smaller than ϕ_0, and so he should not be retained for reasons already discussed.

The case where effort is observable is deal with in a similar way. In this case the IC constraint for effort exertion in the first probationary period is

$$\delta p_T[w_2 - U + \delta\overline{p_T}(w_3 - U)] \geq c, \tag{18}$$

where, as before, $p_T = (1-\gamma)\phi_0 + \lambda(1-\phi_0)$ and $\hat{\phi} = \phi^{y_3}(\phi_0, \bar{e})$, and $\overline{p_T} = \hat{\phi}\alpha + (1-\hat{\phi})\beta$. Let V' be the firm's payoff in the strategy profile under consideration. The IC constraints for the firm's retention decisions are

$$y(\phi^{y_2}(\hat{\phi}, \underline{e}), \underline{e}) - w_3 \geq V' \tag{19}$$

and

$$(1-\delta)[y(\hat{\phi}, \underline{e}) - w_2] + \delta\overline{p_T}\left\{(1-\delta)[y(\phi^{y_2}(\hat{\phi}, \underline{e}), \underline{e}) - w_3] + \delta V'\right\} + \delta(1-\overline{p_T})V' \geq V',$$

which can be rewritten as

$$y(\hat{\phi}, \underline{e}) - w_2 + \delta\overline{p_T}\left\{y(\phi^{y_2}(\hat{\phi}, \underline{e}), \underline{e}) - w_3\right\} \geq [1 + \delta\overline{p_T}]V'. \tag{20}$$

Suppose then, by contradiction, that there exists a pair $(\overline{w}_2, \overline{w}_3)$ satisfying (18) to (20). Now let (w_2, w_3) be such that $w_3 = U$ and $w_2 = \overline{w}_2 + \delta\overline{p_T}(\overline{w}_3 - U)$. Then

$$w_2 - U + \delta\overline{p_T}(w_3 - U) = \overline{w}_2 + \delta\overline{p_T}(\overline{w}_3 - U),$$

and so, since $\overline{w}_3 \geq U$ by limited liability, (w_2, w_3) also satisfies the IC constraints (18) to (20). In particular, it must be that

$$\delta[\phi_0(1-\gamma) + (1-\phi_0)\lambda](w_2 - U) \geq c.$$

By the previous case, however, this is not possible if $\lambda, \delta, \gamma, \phi_0$, and y_3 satisfy the conditions in the above statement. □

We are now ready to state and prove the central result of this paper.

Theorem 3. *Suppose that α and β are such that*

$$\beta \in \left(\frac{3}{7}, \frac{3}{7} + \eta\right) \quad \text{and} \quad \alpha \in \left(\frac{3+\beta}{4}, \min\left\{\frac{6}{7} + \frac{\eta}{4}, 2\beta\right\}\right).$$

Then there exist $\overline{\eta} \in (0,1)$, $\underline{\gamma} \in (0,1)$, $\underline{\lambda} \in (0,1)$, $\overline{\delta} \in (0,1)$, $0 < \underline{\phi_0} < \overline{\phi_0} < 1$, $0 < (1-\alpha)(y_2 - y_1) < \bar{c}$, and $\underline{y_3} > y_2$ such that if $\eta \in (0, \overline{\eta})$, $\gamma < \underline{\gamma}$, $\lambda < \underline{\lambda}$, $\phi_0 \in (\underline{\phi_0}, \overline{\phi_0})$, $\delta \in (\overline{\delta}, 1)$, $y_3 \in (y_2, \underline{y_3})$, and $c \in ((1-\alpha)(y_2 - y_1), \bar{c})$, then all anonymous equilibria of the full commitment are outcome equivalent to the standard up-or-out profile where

$$w_1 = U \quad \textit{and} \quad w_2 = U + \frac{c}{\delta(1+\delta)[\phi_0(1-\gamma) + (1-\phi_0)\lambda]}. \tag{21}$$

Proof: First notice
$$2\beta > \frac{3+\beta}{4} \quad \Rightarrow \quad \beta > \frac{3}{7},$$
and so it is necessary to have $\beta > 3/7$ in order for the interval where α must lie to be well-defined. Moreover, $(3+3/7+\eta)/4 = 6/7+\eta/4$, and so this interval is non-empty. Suppose, from now on, that $\eta < 1/14$. Then
$$\beta > \frac{3}{7} \;\Rightarrow\; \frac{3+\beta}{4} > \beta \;\Rightarrow \alpha \in (\beta, 2\beta) \;\Rightarrow\; (\alpha-\beta)^2 < \beta^2 < \frac{1}{4},$$
and that
$$1-\alpha > \frac{1}{8} \quad \text{and} \quad \alpha+\beta \geq \frac{3+5\beta}{4} > \frac{9}{7} \Rightarrow 2(1-\alpha)(\alpha+\beta) > \frac{1}{4}\cdot\frac{9}{7}.$$
Hence α and β are such that
$$\alpha > \frac{3+\beta}{4}\,, \quad (\alpha-\beta)^2 \leq 2(1-\alpha)(\alpha+\beta)\,, \quad \text{and} \quad \alpha < 2\beta. \tag{22}$$

Consider now the anonymous strategy profile σ where: (*i*) The firm always offers the contract $C = \{w_1', T_1 = 1, w_2', T_2 = 1, w_3', T_3 = 1\}$ to an age 1 worker; (*ii*) If ϕ is the firm's belief about a worker at the end of his first probationary period, retain him if, and only if, $\phi \geq \phi_2$; (*iii*) Suppose a worker was retained at the end of the first probationary period, and let now ϕ be the firm's belief about him at the end of his second probationary period. Retain the worker if, and only if, $\phi \geq \phi_3$; (*iv*) A worker never exerts effort when employed.

The fact that α and β satisfy the three conditions in (22) implies that there exist $\overline{\delta}, \underline{\lambda}, \underline{\gamma} \in (0,1)$, $0 < \underline{\phi_0} < \overline{\phi_0} < 1$, $0 < (1-\alpha)(y_2-y_1) < \overline{c}$, and $\underline{y_3} > y_2$ such that if $\gamma < \underline{\gamma}$, $\lambda < \underline{\lambda}$, $\delta > \overline{\delta}$, $\phi_0 \in (\underline{\phi_0}, \overline{\phi_0})$, $c \in ((1-\alpha)(y_2-y_1), \overline{c})$, and $y_3 \in (y_2, \underline{y_3})$, then the only anonymous strategy profiles that are feasible are either a standard up-or-out profile or a strategy profile that is outcome equivalent to σ.[17] It is obvious that in order for σ to be an equilibrium (and any other strategy profile that is outcome equivalent to it), we must have $w_1' = w_2' = w_3' = U$. In the same way, we must have w_1 and w_2 in a standard up-or-out profile given by (21).

Now observe that
$$\alpha(1-\alpha) < 2\beta\left(1 - \frac{3+\beta}{4}\right) = \frac{\beta(1-\beta)}{2} < \beta(1-\beta).$$

Hence, as a consequence of Theorem 1 in the previous section, the firm's payoff from σ, or any other strategy profile outcome equivalent to it, is
$$V' = \frac{y(\phi_0, \underline{e}) - U + \delta p_1 \left\{y(\phi_1, \underline{e}) - U + \delta p_2[y(\phi_2, \underline{e}) - U]\right\}}{1+\delta p_1(1+\delta p_2)},$$

[17]These strategy profiles differ from σ in how the workers behave off the equilibrium path.

where $p_1 = \phi_0\alpha + (1-\phi_0)\beta$, $\phi_1 = \phi^{y_2}(\phi_0, \underline{e})$, $p_2 = \phi_1\alpha + (1-\phi_1)\beta$, and $\phi_2 = \phi^{y_2}(\phi_1, \underline{e})$. Straightforward algebra shows that

$$V' = \underline{y} - U + \frac{1+\delta\alpha+\delta^2\alpha^2}{1+\delta\phi_0[\alpha+\beta+\delta(\alpha^2+\beta^2)]}\phi_0(\alpha-\beta)(y_2-y_1),$$

where $\underline{y} = \beta y_2 + (1-\beta)y_1$. Therefore, in the limiting case ($\lambda = \gamma = 0$, $\delta = 1$, $\phi_0 = 1/2$ and $y_3 = y_2$) this payoff is

$$V' = \underline{y} - U + (\alpha-\beta)(y_2-y_1)\underbrace{\left\{\frac{1+\alpha+\alpha^2}{2+\alpha+\beta+\alpha^2+\beta^2}\right\}}_{\mu}.$$

The payoff to the firm from the standard up-or-out profile under consideration is

$$V = \frac{y(\phi_0, \overline{e}) - U - c + \delta(1+\delta)p_T[y(\hat{\phi}, \underline{e}) - U]}{1+\delta(1+\delta)p_T},$$

which in the limiting case becomes

$$V = \underline{y} - U + \frac{3}{4}(\alpha-\beta)(y_2-y_1) + \frac{1}{4}(1-\alpha)(y_2-y_1) - \frac{c}{2}.$$

Hence, still in the limiting case,

$$V - V' = (\alpha-\beta)(y_2-y_1)\left(\frac{3}{4}-\mu\right) + \frac{1}{4}(1-\alpha)(y_2-y_1) - \frac{c}{2}.$$

Since $c > (1-\alpha)(y_2-y_1)$, as effort is inefficient, $V - V' > 0$ if, and only if,

$$(\alpha-\beta)\left(\frac{3}{4}-\mu\right) > \frac{1-\alpha}{4}. \tag{23}$$

Suppose $\beta = \frac{3}{7}$ and $\alpha = (3+\beta)/4 = \frac{6}{7}$. Then $\mu = \frac{97}{216} < \frac{1}{2}$, and so (23) holds if

$$\frac{\alpha-\beta}{4} > \frac{1-\alpha}{4} \quad \Leftrightarrow \quad \alpha > \frac{1+\beta}{2} \quad \Leftarrow \quad \alpha = \frac{3+\beta}{4}.$$

We can then conclude that$V > V'$ for η sufficiently small. Since, from Theorem 2, we know that an equilibrium in anonymous strategies exists, we have the desired result. □

5 Conclusion

The paper has rationalized the use of up-or-out contracts in an environment in which a worker's ability is unobserved to both a firm and the worker. Information about ability is acquired by observing the worker's output over time. As a difference with respect to standard experimentation problems, the information generated is affected by the worker's choice of effort. In particular, the likelihood of high output increases in the effort a worker exerts on the job. We have shown that, when prior information about ability is most diffuse, a firm benefits by offering an up-or-out contract, if the increase in the informativeness of output is sufficiently large when the worker exerts effort, as compared to when he chooses no effort.

An issue of interest in this framework is the extent to which commitment on post-retention compensation can be relaxed. In this case, if employment outcomes are to some extent observable to outside labor market participants, a worker's incentive to effort exertion can derive from the wage bidding triggered by a firm's retention decision, at the end of the probationary period. Modelling outside labor market competition in the context of the up-or-out contract game will is the specific object of present and future research.

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