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Inflation, Prices, and Information in Competitive Search

Miquel Faig* and Belén Jerez**

Abstract

Inflation, as a tax on money, gives buyers an incentive to reduce money balances. Sellers are aware of this incentive and try to attract buyers by announcing price offers that reduce the need for buyers to carry precautionary balances. We examine the effect of inflation on equilibrium price offers and associated trades in a competitive search environment where buyers experience preference shocks after they are already matched with a seller. With full information, the equilibrium price structure consist of a single flat fee applied equally to all buyers. If buyer preferences are private information, incentive compatibility forces sellers to charge more to buyers who purchase larger quantities. However, as inflation rises, price schedules become relatively flat. The equilibrium is efficient at the Friedman rule and inflation reduces welfare both with full and private information. With full information, inflation reduces output for all buyer types. With private information, inflation reallocates output from buyers with a high desire to consume to buyers with a low desire to do so.

* Department of Economics, University of Toronto, 150 St. George Street, Toronto, Canada, M5S 3G7. E-mail: mfaig@chass.utoronto.ca

**Departamento de Economía, Universidad Carlos III de Madrid, 28903 Getafe, Spain. E-mail: mjerez@eco.uc3m.es

1 Introduction

Many accounts stress that one of the major consequences of high inflation is that individuals end up buying goods even when they have little appetite for them while they are liquidity constrained when they desire to make a large purchase. For example, Willy Derkow, who was a student during the time of the German hyperinflation, remembered in 1975:¹ “As soon as you caught one (bundle of notes) you made a dash for the nearest shop and bought anything...You very often bought things you did not need.” With lower inflation, this effect might not so easily noticeable to a casual observer, but it is potentially an important adverse effect of inflation. In this paper, we advance a model to capture this effect.

In our model, goods are traded in a competitive search environment. This environment serves our purpose because it combines trade frictions with efficient bilateral trades. The existence of trade frictions is essential to capture the cost of inflation mentioned above, which implies that consumers end up with different marginal rates of substitution. The efficiency of bilateral trades is a desirable modeling strategy because it avoids the inefficient outcomes we seek to model being the result of an inferior trade mechanism.²

In our model, buyers experience preference shocks not only after deciding the demand for money but also after being matched with a seller. This timing is important for our results. First, it gives people an incentive to carry precautionary balances to face the uncertainty of expenditure needs. As people economize on precautionary balances to avoid the inflation tax, they face possible liquidity constraints. Second, each seller serves a potential clientele of diverse buyers. Hence, it opens the possibility of cross-subsidies across different buyer types, so the provision of large quantity of goods to individuals with a low appetite for them is a possible equilibrium outcome.

The main predictions of our model can be summarized as follows. Inflation gives buyers an incentive to reduce money balances. Aware of this incentive, sellers attract buyers by posting price offers that reduce the money balances that buyers need to carry. To this

¹See www.johndclare.net/Weimar_hyperinflation.htm.

²As shown by Rocheteau and Wright (2005), competitive search achieves a first best outcome under the Friedman role, while this is not the case with Nash bargaining or perfect competition. See Kiyotaki and Wright (1989) for a seminal contribution on the search theoretic foundations of money.

end, the posted price offers must avoid the uncertainty of payments and hence reduce the need to carry precautionary balances. With full information, the equilibrium price offers consist of a flat fee which is independent of the quantity purchased by a buyer. As a result, buyers optimally choose an amount of money equal to the flat fee, so they avoid carrying precautionary balances. With private information of preference shocks, incentive compatibility forces sellers to charge buyers a payment which is increasing with the quantity purchased, so a flat fee is not an equilibrium outcome. However, as inflation rises, price schedules become relatively flat to reduce the uncertainty of payments. These flat price schedules imply that buyers have an incentive to purchase relatively large amounts as long as they are not liquidity constrained (have little appetite for goods). Meanwhile, when buyers have a large appetite for goods, they face binding liquidity constraints. Therefore, inflation reallocates output from buyers with a high desire to consume to buyers with a low desire to do so.

The idea that inflation provides incentives to change trading arrangements in order to avoid idle money balances is also found in two recent papers. In Faig and Huangfu (2004), inflation provides an incentive to market-makers to intermediate between buyers and sellers with the objective of eliminating idle money balances. In Berentsen, Camera, and Waller (2004) inflation provides an incentive to banks to do a similar intermediation. In our model, there is no intermediation between buyers and sellers from any third party. Moreover, the idea that inflation relocates output from the people with a high willingness to pay to people less inclined to do so is not present in these papers.

The extension of competitive search to allow for the private information of preference shocks follows our earlier work in Faig and Jerez (2004) (see also Shimer, 2004). This natural extension is a novelty in monetary search models and, as stated above, it has important economic implications.

In a companion paper (Faig and Jerez, 2005), we argue that the precautionary demand for money explains not only the low velocity of circulation of money in the United States, but also its interest elasticity. The model in that paper has a different timing of shocks than the present contribution. In that paper, the preference shocks are realized after the acquisition of money but prior to matching. As a result, sellers are able to post price offers that target

particular buyer types. In competitive search equilibrium, buyers are then separated in different submarkets according to their type, which eliminates the cross-subsidies emphasized here.

The structure of the paper is as follows. Section 2 describes the environment. Section 3 describes the buyer-seller choice and the financial decisions. Sections 4 and 5 characterize the competitive search equilibrium with full and private information, respectively. Section 6 concludes. The proofs are in the Appendix.

2 The Environment

The economy consists of a measure one of individuals. Individuals live in a large number of symmetric villages.³ The members of each village are ex ante identical. They all produce a perishable good specific to the village and consume the goods produced in all villages except for their own. Hence, individuals must trade outside their village to consume.

Time is a discrete, infinite sequence of days. Each morning an individual must choose to be either a buyer or a seller in the goods market that convenes later in the day. Within a village some individuals will be buyers and others will be sellers each day. However, over time individuals will alternate between these two roles.

Individuals seek to maximize their expected lifetime utility:

$$E \sum_{t=0}^{\infty} \beta^t \mathcal{U}(\varepsilon, q_t^b, q_t^s), \quad (1)$$

where

$$\mathcal{U}(\varepsilon, q^b, q^s) = \varepsilon U(q^b) - C(q^s) \quad (2)$$

is the one-period utility function and $\beta \in (0, 1)$ is the discount factor. The one-period utility depends on the quantity consumed q^b if the individual chooses to be a buyer, and on the quantity produced q^s if he chooses to be a seller. It also depends on an idiosyncratic preference shock ε which affects the utility of consumption $\varepsilon U(q^b)$, but does not affect the disutility of production $C(q^s)$. The preference shock is uniformly distributed in the interval

³This environment renders a tractable distribution of money holdings. See Faig (2004) for its relationship with other devices proposed by Shi (1997) and Lagos and Wright (2005) to achieve a similar outcome.

$[1, \bar{\varepsilon}]$, independent across time, and drawn in such a way that the Law of Large Numbers holds across individuals. The cumulative distribution function is then

$$F(\varepsilon) = \varphi(\varepsilon - 1), \quad (3)$$

where φ represents the constant density:

$$\varphi = \frac{1}{\bar{\varepsilon} - 1}. \quad (4)$$

Both U and C are continuously differentiable and increasing. Also, U is strictly concave and C is convex, with $U(0) = C(0) = 0$, and $U'(0) = \infty$. Finally, there is a maximum quantity q^{\max} that the individual can produce each day which satisfies $\bar{\varepsilon}U(q^{\max}) \leq C(q^{\max})$.

Money is an intrinsically useless, perfectly divisible, and storable asset. Units of money are called dollars. The supply of money grows at a constant factor $\gamma > \beta$, so

$$M_{+1} = \gamma M, \quad (5)$$

where M is the quantity of money per individual.⁴ Each day new money is injected via a lump-sum transfer τ common to all individuals. For money to grow at the rate γ , this transfer must satisfy:

$$\tau = (\gamma - 1) M. \quad (6)$$

Each day goods are traded in a decentralized market where buyers and sellers from different villages meet bilaterally. In this market, buyers and sellers search for trading opportunities and the search process competitive (as in Moen (1997) and Shimer (1996)). Prior to the trading process, each seller simultaneously posts an offer, which is a contract detailing the terms at which they commit to trade. Then buyers observe all the posted offers and direct their search towards the sellers posting the most attractive offer (possibly randomizing over offers for which they are indifferent). The set of sellers posting the same offer and the set of buyers directing their search towards them form a submarket. In each submarket buyers and sellers from different villages meet randomly. We assume that individuals experience

⁴For simplicity, the subscript t is omitted in most expressions of the paper, so, for example, M stands for M_t and M_{+1} stands for M_{t+1} .

one match and the short-side of the market is always served.⁵ That is, the probability that a buyer meets a seller in a submarket is

$$\pi^b(\alpha) = \min(1, \alpha), \quad (7)$$

where α is the ratio of sellers over buyers in that submarket. Similarly, the probability that a seller meets a buyer is

$$\pi^s(\alpha) = \min(1, \alpha^{-1}). \quad (8)$$

Finally, when a buyer and a seller meet in a submarket they trade according to the specified offer.

In the decentralized goods market individuals are anonymous and enforcement is limited. This combined with the absence a double coincidence of wants (implied by the ex-ante choice of trading roles) makes money essential (see Kocherlakota (1989)). However, inside a village financial contracts are enforceable. In particular, in each village there is a centralized credit market where a one-period risk-free bond is traded. There is also a centralized insurance market where individuals can insure against their idiosyncratic risks. As it will become apparent, these two centralized markets exhaust the gains from trade inside a village.

The village structure we adopt in this paper allows for a coherent coexistence of money and financial assets. Moreover, the ability of individuals to rebalance their portfolio in their village renders a tractable distribution of money balances. As discussed in Faig (2004), this role is intimately related to the roles played by large households in Shi (1997) and the centralized markets for goods in Lagos and Wright (2005). We adopt the village structure because it proves very useful to our goals.

A typical day proceeds as follows (see Table 1). In the morning, centralized financial markets are open in each village. During this time, financial contracts from the previous day are settled. The government hands out monetary transfers that increase the money supply. Individuals decide whether to be buyers or sellers. They then adjust their holdings of bonds and money, and purchase insurance if they wish. At noon financial markets close and the goods market opens. The competitive search process starts and submarkets are formed. When a buyer and a seller meet in a submarket, the buyer learns her valuation for

⁵As we shall show, this matching technology implies that in equilibrium $\alpha = \pi^b = \pi^s = 1$ in all submarkets.

the seller's good (ε is realized) and the agents trade according to the pre-specified offer. As a result of trade, sellers produce, buyers consume, and money changes hands from buyers to sellers.

Table 1

MORNING			AFTERNOON			
Financial markets are open			Goods market is open			
Previous financial claims Settled	Choice buyer-seller	Choice of bonds, money, insurance	Sellers post offers	Buyers choose among offers	Realization preference shock	Traders meet and trade

Our equilibrium concept combines perfect competition in all centralized financial markets with competitive search in the decentralized goods market. In equilibrium, individuals make optimal choices in the environment where they live. This environment includes a sequence of nominal interest rates and insurance premia, and a sequence of conditions in the goods market to be detailed below (essentially the reservation surpluses of other traders). Individuals have rational expectations about the future conditions of this environment. We focus on symmetric and stationary equilibria where all individuals follow identical strategies and real allocations are constant over time.

To characterize an equilibrium, we adopt the following strategy. First, we describe the buyer-seller choice and the financial decisions of a representative individual given the equilibrium nominal interest rates and insurance premia, as well as some conjectures about the conditions in the afternoon goods market. Then, we characterize the conditions in the goods market in a competitive search equilibrium. Finally, we show that these conditions satisfy our former conjecture. A formal definition of an equilibrium is given at the end of Section 4.

3 Buyer-Seller Choice and Financial Decisions

Consider an individual facing the following environment.

In the credit market, the equilibrium nominal interest rate is:

$$i = \frac{\gamma - \beta}{\beta}, \quad (9)$$

where γ is the growth factor of the money supply and β is the subjective discount factor. Since good prices are proportional to M , which grows at the factor γ , the real interest rate is then equal to the subjective discount rate: $\beta^{-1} - 1$.

In the insurance market, the equilibrium insurance premia are actuarially fair. An individual that decides to be a buyer can purchase an insurance contract which delivers μ_ε^b dollars next day contingent on experiencing a shock ε in the afternoon. The fair premium $\tilde{\mu}^b$ of such a contract is $\tilde{\mu}^b = \int_1^\varepsilon \mu_\varepsilon^b dF(\varepsilon)$. Analogously, the seller can insure against the type of buyer it meets in the goods market. In our environment, there is no need for insuring risks on meeting a trader or not because such risks vanish in equilibrium (all individuals trade with probability one).

We make the conjecture that the goods market has a unique active submarket in equilibrium where all individuals trade. The ratio of buyers over sellers is α . The terms of trade are contingent of the buyer's valuation ε (or type) and are given by $\{q_\varepsilon, d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ where q_ε is the quantity and d_ε is the total payment in dollars of a type- ε buyer.⁶ Since the payments d_ε change over time as the money supply grows, the terms of trade may also be described by $\{q_\varepsilon, z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ where z_ε obeys:

$$z_\varepsilon = \frac{\beta d_\varepsilon}{M_{+1}}. \quad (10)$$

Here z_ε are real payments in next day utils. In a stationary equilibrium the pairs $(q_\varepsilon, z_\varepsilon)$ are time invariant.

Prior to all financial choices, each morning the individual chooses the trading role that yields maximal utility. The value function V of the individual at the beginning of a day then obeys:

$$V\left(\frac{A}{M}\right) = \max\left\{V^b\left(\frac{A}{M}\right), V^s\left(\frac{A}{M}\right)\right\}; \quad (11)$$

where A is the initial wealth in dollars, and V^b and V^s are the value functions conditional on being a buyer or a seller during the day, respectively. The money supply is used to deflate

⁶Since there is a large number of villages, each with a continuum of individuals, there is a large number of buyers of each type ε in a symmetric equilibrium.

nominal quantities. This deflator is appropriate because goods prices increase proportionately with M (see (5) and (6)). The ratio A/M can be interpreted as initial real wealth and is denoted by a .

While financial markets are open, the individual reallocates wealth and may also purchase insurance. Conditional on being a buyer the individual chooses the demands for money, m^b , bonds, b^b , and the insurance coverages, $\{\mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$, to solve:

$$V^b(a) = \max_{m^b, b^b, \{\mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha) [\varepsilon U(q_\varepsilon) + \beta V(a_{+1}^{b\varepsilon})] + [1 - \pi^b(\alpha)] \beta V(a_{+1}^{b0}) \right\} dF(\varepsilon) \quad (12)$$

subject to

$$a_{+1}^{b\varepsilon} = \frac{m^b + b^b(1+i) + \mu_\varepsilon^b - \tilde{\mu}^b + \tau - d_\varepsilon}{M_{+1}}, \quad (13)$$

$$a_{+1}^{b0} = \frac{m^b + b^b(1+i) - \tilde{\mu}^b + \tau}{M_{+1}} \quad (14)$$

$$a = \frac{m^b + b^b}{M}, \text{ and} \quad (15)$$

$$m^b \geq d_\varepsilon \text{ for all } \varepsilon \in [1, \bar{\varepsilon}]. \quad (16)$$

The buyer meets a seller with probability $\pi^b(\alpha)$. The preference shock ε is then realized and the buyer purchases q_ε for d_ε dollars. In this event, next period's real wealth $a_{+1}^{b\varepsilon}$ is given by (13). If the buyer does not meet a seller, she buys nothing and next period's real wealth a_{+1}^{b0} is given by (14). The choice of how to allocate wealth between money m^b and bonds b^b must satisfy the budget constraint (15). In addition, m^b must satisfy (16) since the buyer must carry enough money to face all contingent payments.

Conditional on being a seller the individual chooses the demands for money m^s and bonds b^s to solve:

$$V^s(a) = \max_{m^s, b^s} \int_1^{\bar{\varepsilon}} \left\{ \pi^s(\alpha) [\beta V(a_{+1}^{s\varepsilon}) - C(q_\varepsilon)] + [1 - \pi^s(\alpha)] \beta V(a_{+1}^{s0}) \right\} dF(\varepsilon) \quad (17)$$

subject to

$$a_{+1}^{s\varepsilon} = \frac{m^s + b^s(1+i) + \mu_\varepsilon^s - \tilde{\mu}^s + \tau + d_\varepsilon}{M_{+1}}, \quad (18)$$

$$a_{+1}^{s0} = \frac{m^s + b^s(1+i) - \tilde{\mu}^s + \tau}{M_{+1}}, \quad (19)$$

$$a = \frac{m^s + b^s}{M}, \text{ and} \quad (20)$$

$$m^s \geq 0. \quad (21)$$

The seller meets a buyer with probability $\pi^s(\alpha)$ and, contingent on the buyer's type, sells q_ε for d_ε dollars. If the seller does not meet a buyer he sells nothing. Next period real wealth in each event is given by (18) and (19). The budget constraint (20) must be satisfied and money cannot be negative, (21).

In addition to all constraints specified above, the individual faces an endogenous lower bound on next period real wealth because he or she must be able to repay the amounts borrowed with probability one without reliance to unbounded borrowing (No-Ponzi game condition):

$$a_{+1} \geq a_{\min} \text{ with probability one.} \quad (22)$$

We denote as a_{+1} is the stochastic real wealth for next period, which depends on the choice of being a buyer or a seller, the realization of ε , and the trading match. The endogenous lower bound a_{\min} is equal to minus the present discounted value of the maximum guaranteed income the individual can obtain as a seller.

The optimization program described in equations (11) to (??) is easily solved once the value function V is known. The value function V is a well defined function of a that can be characterized using standard recursive methods. Also, V is concave with a linear segment as stated in the following proposition and proved in the Appendix.

Proposition 1: *There is an interval $[\underline{a}, \bar{a}] \subset [a_{\min}, \infty)$ where the equilibrium value function V takes the linear form*

$$V(a) = v_0 + a. \quad (23)$$

where v_0 is a term independent of a . Outside this interval, V is strictly concave and continuously differentiable. Finally, the interval $[\underline{a}, \bar{a}]$ is absorbing, that is $a \in [\underline{a}, \bar{a}]$ implies $a_{+1} \in [\underline{a}, \bar{a}]$ with probability one.

The linear interval of V is due to the endogenous choice of the trading role individuals make each day. Intuitively, if an individual is not rich enough to be a buyer forever and not so poor to have to be a seller at perpetuity, then the individual will alternate between being a buyer and a seller. As the individual does so, wealth does not affect the quantities consumed or produced, instead it affects how often and how early the individual consumes or produces. Since utility is linear on the times and the timing an individual consumes and produces, the value function is linear.

The property that the interval $[a, \bar{a}]$ is absorbing simplifies the model dramatically. Assuming that all individuals have initial wealth in the interval $[a, \bar{a}]$, as we assume from now on, the behavior of all buyers and all sellers is independent from their wealth. Therefore, there is no incentive to create submarkets that cater to individuals of different wealth and the distributions of money holdings are easily characterized.

The optimal demands for money follow from the fact that money earns not interest but bonds earn $i > 0$. This implies that it is not optimal to carry money balances that are never used. Therefore, m^b is equal to the highest contingent payment: $m^b = \max \{d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ and $m^s = 0$. Using these optimal demands for money, (23), and $a_{+1} \in [a, \bar{a}]$ with probability one, the value functions of the buyer (12) and the seller (17) simplify into:

$$V^b(a) = S^b + \beta \left(v_0 + \frac{\gamma - 1}{\gamma} \right) + a \text{ and} \quad (24)$$

$$V^s(a) = S^s + \beta \left(v_0 + \frac{\gamma - 1}{\gamma} \right) + a. \quad (25)$$

These value functions differ only in the first term. This term represents the expected trading surpluses of buyers and sellers in the afternoon goods market:

$$S^b = \int_1^{\bar{\varepsilon}} \pi^b(\alpha_\varepsilon) [\varepsilon U(q_\varepsilon) - z_\varepsilon] dF(\varepsilon) - im, \text{ and} \quad (26)$$

$$S^s = \int_1^{\bar{\varepsilon}} \pi^s(\alpha_\varepsilon) [z_\varepsilon - C(q_\varepsilon)] dF(\varepsilon). \quad (27)$$

In (26), we define m to be the real money in next day utils: $m \equiv \beta m^b / M_{+1}$. Since buyers carry only enough money to make the highest contingent payment, we have

$$m \equiv \beta m^b / M_{+1} = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]} \cdot \quad (28)$$

Note that the insurance coverages are missing from (26). As long as $a \in [\underline{a}, \bar{a}]$ the individual is indifferent between purchasing insurance or not. The only role played by insurance markets is to ensure that wealth does not drift out of the interval $[\underline{a}, \bar{a}]$. This role is only important if buyers purchase nothing for low realizations of ε . If buyers purchase positive amounts for all realizations of ε then, in general, insurance markets are redundant. In this case, the individual prevents a_{+1} from drifting below \underline{a} by choosing to be a seller and prevents it from drifting above \bar{a} by choosing to be a buyer.

4 Competitive Search with Full Information

In this section we characterize a competitive search equilibrium in the goods market given the morning financial decisions. We show that the conjecture in Section 3 is satisfied. Then we characterize a symmetric monetary stationary equilibrium where all individuals have initial wealth $a \in [\underline{a}, \bar{a}]$.

When the goods market opens sellers post their offers. An offer is a schedule $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$, by means of which a seller commits to sell q_ε units of output in exchange of a real payment z_ε in the event of being matched with a buyer of type ε .⁷ All individuals have rational expectations regarding the number of buyers that will be attracted by each offer, and thus about the relative proportion of buyers and sellers that will trade in each submarket. In a competitive search equilibrium the offers posted by the sellers must be such that sellers have no incentives to post deviating offers.

Let Ω be the set of all submarkets $[\alpha, \{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}]$ that are formed in equilibrium. A competitive search equilibrium is a set $\{\Omega, \bar{S}^b, \bar{S}^s\}$ such that

1. All buyers attain the same expected surplus \bar{S}^b .
2. All sellers attain the same expected surplus \bar{S}^s .

⁷We could allow for offers which are contingent both on the type ε and the wealth a of the buyer. However, from the sellers' view point all buyers of a given type ε are identical even if their wealth is different because their expected surplus (26) and money balances (28) are independent of a . Hence, restricting to offers which are only contingent on ε is without loss of generality.

3. The expected surpluses of buyers and sellers are identical: $\bar{S}^b = \bar{S}^s$

4. Each $\omega \in \Omega$ solves the following program:

$$\bar{S}^b = \max_{[\alpha, \{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}] } \int_1^{\bar{\varepsilon}} \{\pi^b(\alpha) [\varepsilon U(q_\varepsilon) - z_\varepsilon]\} dF(\varepsilon) - im \quad (29)$$

subject to

$$m = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}, \quad (30)$$

$$\int_1^{\bar{\varepsilon}} \{\pi^s(\alpha) [z_\varepsilon - C(q_\varepsilon)]\} dF(\varepsilon) = \bar{S}^s, \text{ and} \quad (31)$$

Buyers ex ante identical and they are free to choose the submarket where they participate, so they must attain the same expected surplus. The same is true for sellers. Also, for trade to occur in equilibrium there must be buyers and sellers present in that submarket, so individuals must be indifferent between the two trading roles. Optimal behavior and competition by sellers lead to condition 4. This condition says that buyers choose among submarkets in order to maximize their expected surplus subject to their cash constraint and the constraint that sellers receive a fixed expected surplus \bar{S}^s . Sellers never post deviating offers that imply a lower expected surplus because they can attain \bar{S}^s in the current submarket.⁸ If a seller tries to post an offer that attracts buyers and yields a higher expected surplus, other sellers would profitably undercut this offer (e.g. by offering those buyers the same quantity for a slightly lower payment). The cash constraint (30) ensures that the buyer is able to pay for the good for any realization of ε .⁹

Program (29) to (30) implies that in equilibrium the total expected surplus from a match must be maximal subject to the cash constraint. But then buyers and sellers must trade with probability one in any active submarket:

$$\alpha = \pi^b(\alpha) = \pi^s(\alpha) = 1. \quad (32)$$

⁸Since individuals are infinitesimal in the market, they take as given the expected surplus of other individuals.

⁹We assume that seller's offers require buyers to pay for the good before ε is realized. If buyers cannot be forced to pay before they learn their type program (29) to (30) is further restricted by an individual rationality constraint that buyers must be willing to make the corresponding payments after they know their type.

The sellers' expected surplus (31) depends on the buyer's average payment, but it does not depend on higher moments of the distribution of $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$. In contrast, for a given average payment, a buyer prefers a smooth distribution of $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ because the opportunity cost of holding money depends on the maximum payment. Therefore, equilibrium payments are uniform:

$$z_\varepsilon = m \text{ for } \varepsilon \in [1, \bar{\varepsilon}]. \quad (33)$$

Substituting (33) and (32) into (31) yields

$$m = \bar{S}^s + \int_1^{\bar{\varepsilon}} C(q_\varepsilon) dF(\varepsilon). \quad (34)$$

Using (33) to (34), program (29) to (30) simplifies to

$$\bar{S}^b = \max_{\{q_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - (1+i)C(q_\varepsilon)] dF(\varepsilon) - (1+i)\bar{S}^s. \quad (35)$$

The equilibrium quantities are then given by the first order condition of this program:

$$\varepsilon U'(q_\varepsilon) = (1+i)C'(q_\varepsilon) \text{ for } \varepsilon \in [1, \bar{\varepsilon}]. \quad (36)$$

To complete the characterization of a competitive search equilibrium, it remains is to determine \bar{S}^s . Since buyers and sellers attain the same expected surplus, (34) and (35) imply:

$$m = \frac{1}{2+i} \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) + C(q_\varepsilon)] dF(\varepsilon). \quad (37)$$

We are ready to define an equilibrium of the monetary economy:

A **monetary stationary equilibrium** is a vector of real numbers $(i, \alpha, m, \bar{S}^s)$ and a set of real functions $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfy the system of equations: (9), (33), (32), (34), (36), and (37). This equilibrium is consistent with the environment conjectured in Section 3. In particular, since the solution to program (29) to (30) is unique, there at most one active submarket in equilibrium.

We have shown that optimal trading offers that minimize the opportunity cost of money balances by having z_ε identical for all ε . Buyers optimally choose an amount of money m equal to the uniform payment and spend all their cash. The welfare effects of inflation are

captured by equations (36), (34) and (37), together with the equation that determines the equilibrium nominal interest rate (9). At the Friedman rule, $i \rightarrow 0$, the quantities of output traded are efficient. The convexity of C and concavity of U imply that q_ε is an increasing function of ε , so high types purchase more output than low types. As inflation rises the opportunity cost of holding money increases inducing buyers to reduce their money holdings. Sellers adjust by reducing their fees. But buyers anyway respond by purchasing lower quantities in all trading meetings and carrying too little money (so they face binding liquidity constraints when faced with abnormally good trading opportunities). That is, q_ε is a decreasing function of i for all ε . These reductions of output relative to the efficient quantities represent the welfare cost of inflation.

The properties of the demand for money and the welfare cost of inflation are essentially those of a standard cash-in-advance model. Higher nominal interest rates reduce both the demand for money and the output traded for all buyer types because in (36) the cost of goods is multiplied by the factor $(1 + i)$ as in cash-in-advance models.

The equilibrium pricing structure is only implementable if preference shocks are observed by the seller. has to undesirable properties. With a uniform payment higher types receive more output and yet pay the same. Unless shocks are observed by the seller, buyers then have an obvious incentive to lie and say they have the highest type $\bar{\varepsilon}$. In the next section, we consider the case that preference shocks are private information.

5 Competitive Search with Private Information

In this section, we characterize a competitive search equilibrium when shocks are privately observed by buyers. In this case, the offers posted by sellers must be incentive compatible. That is, offers must give buyers an incentive to truthfully reveal their type.¹⁰ Program (29)

¹⁰If shocks are not observable in the village or origin insurance may not exist. This is irrelevant for the characterization of an equilibrium as we define it because V is affine in the relevant segment. However, the absence of insurance changes the values of \underline{a} and \bar{a} in (53) and so the set of parameter values for which an equilibrium exists.

to (31) is then further restricted to satisfy the incentive compatibility constraint:¹¹

$$\varepsilon' \in \arg \max_{\varepsilon \in [1, \bar{\varepsilon}]} [\varepsilon' U(q_\varepsilon) - z_\varepsilon], \text{ for all } \varepsilon' \in [1, \bar{\varepsilon}] \quad (38)$$

As is standard, we restate the incentive compatibility constraint (38) using the following well-known result (see Mas-Colell, Winston and Green, 1995, Proposition 23.D.2).

Let the indirect ex-post trade surplus of a type- ε buyer be defined as

$$v_\varepsilon \equiv \varepsilon U(q_\varepsilon) - z_\varepsilon. \quad (39)$$

A trading offer satisfies the incentive compatibility constraint (38) if and only if q_ε is non-decreasing in ε and v_ε satisfies

$$v_\varepsilon - v_1 = \int_1^\varepsilon \frac{\partial}{\partial x} [xU(q_x) - z_x] dx = \int_1^\varepsilon U(q_x) dx, \text{ for all } \varepsilon \in [1, \bar{\varepsilon}]. \quad (40)$$

Using Lemma 5, (32), and (39), the restricted program can be restated as an optimal control problem:

$$\bar{S}^b = \max_{[m, \{(q_\varepsilon, v_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}] } \int_1^{\bar{\varepsilon}} v_\varepsilon dF(\varepsilon) - im \quad (41)$$

subject to

$$\int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] dF(\varepsilon) = \bar{S}^s, \quad (42)$$

$$\varepsilon U(q_\varepsilon) - v_\varepsilon \leq m \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \quad (43)$$

$$\dot{v}_\varepsilon = U(q_\varepsilon) \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \text{ and} \quad (44)$$

$$q_\varepsilon \text{ is non-decreasing in } \varepsilon. \quad (45)$$

¹¹Formally, an offer $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ is a direct revelation mechanism that is incentive compatible. We could also allow for random direct revelation mechanisms. However, as shown by Maskin and Riley (1984), random direct revelation mechanisms are only optimal if absolute risk aversion decreases with the buyers type. In our environment absolute risk aversion is the same for all types, so random mechanisms are never used in equilibrium. See, however, the competitive search labor model in Shimer (2004).

The control of this problem is q_ε while v_ε is the state variable. The optimal solution is characterized using the Maximum Principle (see the Appendix). The optimal path for the control variable q_ε obeys:

$$\begin{cases} (\varepsilon - \gamma_2) U'(q_\varepsilon) = \gamma_1 C'(q_\varepsilon) & \text{for } \varepsilon \in [1, \hat{\varepsilon}], \quad \text{and} \\ q_\varepsilon = q_{\hat{\varepsilon}} \equiv \hat{q} & \text{for } \varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]; \end{cases} \quad (46)$$

where γ_1 , γ_2 , and $\hat{\varepsilon}$ satisfy:

$$\gamma_1 = \frac{1+i}{1+2i}, \quad (47)$$

$$\gamma_2 = \frac{i}{1+2i}, \quad \text{and} \quad (48)$$

$$\gamma_1 + \frac{\gamma_2}{\bar{\varepsilon}} = \frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \frac{1}{2} \left[1 - \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}} \right)^2 \right]. \quad (49)$$

Here represents break-point shock $\hat{\varepsilon}$ where the cash constraint becomes binding. Combining (47) to (49), we obtain $\hat{\varepsilon}$ as a function of i :

$$\frac{i}{\varphi} \frac{\bar{\varepsilon}}{1+2i} = \frac{(\bar{\varepsilon} - \hat{\varepsilon})^2}{2}. \quad (50)$$

The optimal path for the state variable v_ε is implied by the differential equation (44) for a given initial value v_1 . The initial value v_1 in equilibrium is determined by (42) together with the condition for the coexistence of buyers and sellers in the market: $\bar{S}^s = \bar{S}^b$. The optimal value of m is given by (43) with equality at the break-point $\hat{\varepsilon}$. Finally, the underlying payments $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ are calculated from (39).

A monetary stationary equilibrium is a vector of real numbers $(i, \gamma_1, \gamma_2, \hat{\varepsilon}, \alpha, m, \bar{S}^s, \bar{S}^b)$ and a set of real functions $\{(q_\varepsilon, v_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfy the system of equations: (9), (32), (41), (42), (43) with equality at $\hat{\varepsilon}$, (44), (46), (47), (48), (50), and $\bar{S}^s = \bar{S}^b$.

The equilibrium is implemented if sellers post an increasing non-linear price schedule. For buyers to choose the quantities of output consistent with (46), they must face a price schedule that has the form:

$$Z(q) = \gamma_0 + \gamma_1 C(q) + \gamma_2 U(q), \quad (51)$$

where γ_0 is a constant. That is, buyers pay more for larger quantities. As inflation rises, γ_1 falls and γ_2 increases. Therefore, the equilibrium price schedule becomes more flat. That is, the offers posted by the sellers in equilibrium try to minimize the increase in the cost of idle money balances.

In addition to the price schedule, trading offers must include some additional restrictions. The reason is that in general a buyer facing (51) will not choose a quantity of money that is consistent with the threshold $\hat{\varepsilon}$ in equation (50). Instead, the buyer would carry too much money if $\bar{\varepsilon} > (1 + 2i)\hat{\varepsilon}$, which occurs for high values of $\bar{\varepsilon}$. In this case, trading offers must include a cap on output at \hat{q} . Conversely, the buyer would carry too little money if $\hat{\varepsilon}$ if $\bar{\varepsilon} < (1 + 2i)\hat{\varepsilon}$, which occurs for low values of $\bar{\varepsilon}$. In this case, trading offers must include a restriction on the minimum amount of money that buyers carry (the equilibrium m).

The equilibrium is efficient at the Friedman Rule as in the full information model. That is, as $i \rightarrow 0$ the cash constraint never binds: $\hat{\varepsilon} = \bar{\varepsilon}$. Also, $\gamma_1 = 1$ and $\gamma_2 = 0$, so the quantities traded are efficient. Unlike in the case of full information, money circulates faster as i rises not only because buyers reduce their money balances ($\hat{\varepsilon}$ falls), but also because they increase their purchases when they are not liquidity constrained. That is, an increase in i reduces m and \hat{q} but *increases* q_ε for $\varepsilon \in (1, \hat{\varepsilon})$. This can be shown by applying the Implicit Function Theorem to the system of equations (46) to (48). This application implies that for all $\varepsilon \in [1, \hat{\varepsilon}]$:

$$\frac{dq_\varepsilon}{di} = \frac{\varepsilon - 1}{(1 + i)(1 + 2i)} \frac{U'(q_\varepsilon)}{\gamma_1 C''(q_\varepsilon) - (\varepsilon - \gamma_2) U''(q_\varepsilon)} > 0. \quad (52)$$

Consequently, inflation not only curtails consumption due to lack of liquidity for those buyers with high valuations ($\varepsilon > \hat{\varepsilon}$), but it also increases consumption for those buyers with a low valuations ($\varepsilon < \hat{\varepsilon}$). These deviations from the efficient output quantities represent the welfare cost of inflation. Equations (51) and (52) imply that z_ε is an increasing function of i since $z_\varepsilon = Z(q_\varepsilon)$ for $\varepsilon < \hat{\varepsilon}$. Therefore, as i increases buyers spend a larger fraction of their money balances when they are not liquidity constrained. The increase in the payments z_ε combined with the reduction of real money balances m reduces the fraction of unspent money in the economy.

6 Conclusion

We have provided a model, to capture the popular accounts that during high inflation episodes individuals end up buying goods they care little about while they are liquidity constrained when they have a good trading opportunity. The key elements of our model are the following: competitive search, preference shocks realized after matching, and private information of these shocks. The intuition of our main result goes as follows. Since inflation represents a tax on money balances, sellers attract buyers by posting price offers that reduce the money balances that buyers need to carry. To this end, the posted price offers must avoid the uncertainty of payments. With private information of preference shocks, this uncertainty cannot be completely eliminated because of incentive compatibility constraints. However, as inflation rises, price schedules become relatively flat. These flat price schedules imply that buyers have an incentive to purchase relatively large amounts as long as they are not liquidity constrained. Meanwhile, when buyers have a large appetite for goods, they face binding liquidity constraints. Therefore, inflation reallocates output from individuals with a high desire to consume to individuals with a low desire to do so.

Appendix

Proof Proposition 1

Consider the problem of an individual in the equilibrium of our basic model where all other individuals have value functions (23). These other individuals have initial wealths in the interval $[\underline{a}, \bar{a}]$. Throughout the appendix, we use without further proof the absence of uncertainty in trading opportunities because of efficient matching.

For all finite $a \geq a_{\min}$, the set of feasible time and state contingent policies is non empty. The feasible values of the quantities consumed and produced are bounded. Also, for all the feasible policies the present discounted utility is well defined and finite because \mathcal{U} is a continuous function. Consequently, we can use standard recursive methods to find the value function.

In competitive search, we can recursively characterize the individual optimization problem as follows.¹² The individual chooses to be a buyer or a seller. As a buyer the individual chooses $\left(\{q_\varepsilon^b, z_\varepsilon^b, \mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}, m^b, b^b \right)$, where $\{q_\varepsilon^b, z_\varepsilon^b, \mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ are the set of choices contingent on the realization of their preference shock. As a seller, the individual chooses $\left(\{q_\varepsilon^s, z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]}, m^s, b^s \right)$, where $\{q_\varepsilon^s, z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ is the trading offer posted by the seller. These choices are subject to the constraints (13)-(16), (18)-(21), and (??). Moreover, in the financial markets the individual takes as given the rate of interest and the insurance premia. In the goods market, the individual takes as given the reservation expected trade surpluses of other traders. Therefore, as a seller, the individual must make offers that gives buyers the expected trade surplus they can attain in alternative submarkets: the posted offers must be a subset of $\{q_\varepsilon^s, z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]}$, which satisfies $\int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon^s) - z_\varepsilon^s] dF(\varepsilon) - i \max \{z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]} \geq \bar{S}^b$. As a buyer, the individual acts as if he/she were choosing $\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfies $\int_1^{\bar{\varepsilon}} [z_\varepsilon^b - C(q_\varepsilon^b)] dF(\varepsilon) \geq \bar{S}^s$, because competition among sellers drives offers to be the best possible for the buyers that provide sellers with the trade surplus \bar{S}^s .

Let $\mathcal{C}(a)$ be the space of bonded and continuous functions $f : [a_{\min}, \infty) \rightarrow \mathcal{R}$, with the sup norm. Use the Bellman's equations (12) and (17) together with (11) to define the mapping T of $\mathcal{C}(a)$ onto itself by substituting f for V in the right hand sides of (12) and (17) and

¹²This characterization uses a more general definition of competitive search than the text because it allows the individual to have wealth outside the interval $[\underline{a}, \bar{a}]$.

denoting as $Tf(a)$ the left hand side of (11). The choice variables and constraints of these maximization programs are described in the previous paragraph. For a given a , the set of feasible policies is non-empty, compact-valued, and continuous. The utility function \mathcal{U} is a bounded and continuous on the set of feasible policies, and $0 < \beta < 1$. Therefore, Theorem 4.6 in Stokey and Lucas with Prescott (1989) implies that there is a unique fixed point to the mapping T , which is the value function V .

Let $\mathcal{V}(a)$ be the sup normed space of functions $f : [a_{\min}, \infty) \rightarrow R$ that satisfy (23) for v_0 , \bar{a} , and \underline{a} that satisfy:

$$\begin{aligned} v_0 &= \frac{\bar{S}^s}{1-\beta} + \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}, \\ \bar{a} &= \frac{\int_1^{\bar{\varepsilon}} z_\varepsilon dF(\varepsilon) + im}{1-\beta} - \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}, \text{ and} \\ \underline{a} &= -\frac{\int_1^{\bar{\varepsilon}} z_\varepsilon dF(\varepsilon)}{1-\beta} - \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}; \end{aligned} \tag{53}$$

where i , m , \bar{S}^s , and z_ε satisfy the equilibrium system of equations described in 4. Consider the mapping T defined in the previous paragraph. Since V is concave, it is an optimal policy to fully insure preference shocks (full insurance is strictly optimal if there is a positive probability that $a_{+1} \notin [\underline{a}, \bar{a}]$). In consequence, a_{+1} is not stochastic. Let a_{+1}^b be next period real wealth for an optimal policy conditional on being a buyer. Similarly, let a_{+1}^s be the optimal policy for a seller. If $a_{+1}^b, a_{+1}^s \in [\underline{a}, \bar{a}]$, $TV(a)$ is the maximum of $V^b(a)$ and $V^s(a)$ in equations (24) and (25), so $TV(a)$ is affine and the trade surpluses are those in (26) and (27). The optimal policies of the individual are the equilibrium ones modeled in the main text. Therefore, the individual is indifferent between being a buyer or a seller. This indifference is broken when one policy would lead to $a_{+1} \notin [\underline{a}, \bar{a}]$. In such a case, the strict concavity of V outside the interval $[\underline{a}, \bar{a}]$ implies that it is suboptimal to be a seller if $a_{+1}^s > \bar{a}$. Likewise, it is suboptimal to be a buyer if $a_{+1}^b < \underline{a}$. Consequently, the recursive budgets (13) to (15) and (18) to (20), together with (53), imply that $a_{+1} \in [\underline{a}, \bar{a}]$ if and only if $a \in [\underline{a}, \bar{a}]$. This implies that $TV(a)$ is affine in the interval $[\underline{a}, \bar{a}]$. Equation (25) implies that the constant term of this affine function is the value of v_0 in (53). If $a > \bar{a}$, the optimal policy is to be a buyer. Vice versa, if $a < \underline{a}$, an optimal policy is to be a seller. In both cases, the strict concavity of U and convexity of C imply the strict concavity of $TV(a)$ for $a \notin [\underline{a}, \bar{a}]$. In summary, T maps

$\mathcal{V}(a)$ onto itself. Therefore, the value function V satisfies (23). Finally, since V is concave, \mathcal{U} is continuously differentiable, and the solution is interior, V is continuously differentiable.

Competitive Search Equilibrium with Private Information

In this section, we solve program (41) to (45) in two stages. Stage 1 (Statements 1 to 13) solves for the program for a given the Lagrange multiplier λ associated with constraint (42), and given m and v_1 . Stage 2 (Statements 14 to 18) endogenizes λ , m , and v_1 .

1. Let $\lambda > 1/2$ and $m > -v_1$. The terms of trade in a competitive search equilibrium with private preference shocks solve the following program:¹³

$$J(\lambda, v_1, m) = \max_{\{q_\varepsilon, v_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}} \int_1^{\bar{\varepsilon}} \{v_\varepsilon + \lambda[\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon]\} dF(\varepsilon) \quad (54)$$

subject to

$$\dot{v}_\varepsilon = U(q_\varepsilon), \quad (55)$$

$$z_\varepsilon \equiv \varepsilon U(q_\varepsilon) - v_\varepsilon \leq m, \quad (56)$$

$$q_\varepsilon \geq 0, \text{ and} \quad (57)$$

$$v_1 \text{ given.} \quad (58)$$

2. Program (54) to (58) is a standard optimal control problem with q_ε as the control variable and v_ε as the state variable. A solution to the program exists because the set of feasible paths is non-empty, bounded, and there exists a feasible path for which the objective in (54) is finite. For example, the path $q_\varepsilon = 0$ for all ε and $v_\varepsilon = v_1$ is feasible, and with this path the objective in (54) is finite.
3. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the inequality constraint (57) is binding, that is $q_\varepsilon = 0$ for $\varepsilon \in [a, b]$. Then (55), (56), and $U(0) = 0$ imply that in this interval z_ε is constant and equal to $-v_a \leq -v_1$. Since $a \leq \bar{\varepsilon}$ and $m > -v_1$, constraint (56) is not binding in $[a, b]$. Therefore, constraints (56) and (57) never bind simultaneously.

¹³The constraint q_ε must be a non-decreasing function of ε is omitted for the time being because as it will be seen it is not binding.

4. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the inequality constraint (56) is binding, that is $z_\varepsilon = m$ for $\varepsilon \in [a, b]$. Then Statement 3 implies that in this interval $q_\varepsilon > 0$, so $U(q_\varepsilon) > 0$. Hence, (55) and (56) imply that q_ε is constant in the interval $[a, b]$.
5. Let ϖ_ε denote the co-state variable associated with (55), and ς_ε and ϑ_ε be the Lagrange multipliers associated with (56) and (57) respectively. The Hamiltonian of the program (54) to (58) is:

$$\mathcal{H} = v_\varepsilon \varphi + \lambda [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] \varphi + \varpi_\varepsilon U(q_\varepsilon) + \varsigma_\varepsilon [m - \varepsilon U(q_\varepsilon) + v_\varepsilon] + \vartheta_\varepsilon q_\varepsilon. \quad (59)$$

6. For the values of ε such that (56) is not binding, the Hamiltonian (59) is strictly concave with respect to q_ε (for these values $\varsigma_\varepsilon = 0$) and linear (and so concave) with respect to v_ε . For the values of ε such that (56) is binding, q_ε is a constant (Statement 4). Therefore, the solution to the program (54) to (58) is unique, it is characterized by the first order conditions that result from applying the Maximum Principle, and both q_ε and v_ε are continuous functions of ε .
7. The first order condition with respect to the control variable q_ε is ($\mathcal{H}_{q_\varepsilon} = 0$):

$$(\lambda \varphi - \varsigma_\varepsilon) \varepsilon U'(q_\varepsilon) + \varpi_\varepsilon U'(q_\varepsilon) = \lambda \varphi C'(q_\varepsilon) - \vartheta_\varepsilon. \quad (60)$$

The co-state variable must obey ($\mathcal{H}_{v_\varepsilon} = -\dot{\varpi}_\varepsilon$):

$$\dot{\varpi}_\varepsilon = (\lambda - 1) \varphi - \varsigma_\varepsilon. \quad (61)$$

Finally, the transversality condition implies¹⁴:

$$\varpi_{\bar{\varepsilon}} = 0. \quad (62)$$

Integrating (61) for an interval $[\varepsilon, \bar{\varepsilon}]$ and using (62), the value of the co-state variable ϖ_ε is solved to obtain:

$$\varpi_\varepsilon = (\lambda - 1) \varphi (\varepsilon - \bar{\varepsilon}) + \Sigma_\varepsilon, \quad (63)$$

¹⁴The transversality condition is $\varpi_{\bar{\varepsilon}} v_{\bar{\varepsilon}} = 0$. However, $v_{\bar{\varepsilon}} > 0$ if $v_1 > 0$ given $U(\cdot) \geq 0$ and (55). If $v_1 = 0$ still $v_{\bar{\varepsilon}} > 0$. If $v_{\bar{\varepsilon}} = 0$ then $v_\varepsilon = 0$ for all ε (as v_ε is non-decreasing). But this is impossible since the buyer's expected utility is strictly positive in equilibrium.

where, to simplify the algebraic notation, we use the following definition:

$$\Sigma_\varepsilon \equiv \int_\varepsilon^{\bar{\varepsilon}} \varsigma_u du. \quad (64)$$

Using (63), the first order condition (60) is transformed into:

$$[(2\lambda - 1)\varphi - \varsigma_\varepsilon] \varepsilon U'(q_\varepsilon) = [(\lambda - 1)\varphi\bar{\varepsilon} - \Sigma_\varepsilon] U'(q_\varepsilon) + \lambda\varphi C'(q_\varepsilon) - \vartheta_\varepsilon. \quad (65)$$

8. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the two inequality constraints (56) and (57) are not binding. Then the Kuhn-Tucker Theorem implies $\varsigma_\varepsilon = \vartheta_\varepsilon = 0$ for $\varepsilon \in [a, b]$, so the first order condition (65) simplifies into

$$(\varepsilon - \gamma_2) U'(q_\varepsilon) = \gamma_1 C'(q_\varepsilon) \quad \text{for } \varepsilon \in [a, b], \quad (66)$$

where

$$\gamma_1 = \frac{\lambda}{2\lambda - 1}, \quad \text{and} \quad \gamma_2 = \frac{(\lambda - 1)\bar{\varepsilon} - \Sigma_b \varphi^{-1}}{2\lambda - 1}. \quad (67)$$

Since both $U'(q_\varepsilon)$ and $C'(q_\varepsilon)$ are strictly positive for q_ε strictly positive and $\lambda > 1/2$, (66) can only hold for $\varepsilon > \gamma_2$. The Implicit Function Theorem applied to (66) implies that q_ε is an increasing function of ε in the interval $[a, b]$. This property combined with (55), (56) and $U'(q_\varepsilon) \geq 0$ implies that z_ε is also increasing in the interval $[a, b]$.

9. Combining Statements 3, 4, 6, and 8, z_ε is a non-decreasing continuous function for all $\varepsilon \in [1, \bar{\varepsilon}]$. Therefore, either (56) is never binding, or it is binding in an interval of high values of $\varepsilon : [\hat{\varepsilon}, \bar{\varepsilon}]$. In such an interval, Statement 4 implies that q_ε is positive and constant: $q_\varepsilon = \hat{q}$ for $\varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]$.
10. Combining Statements 3, 6, 8, and 9, q_ε is a non-decreasing continuous function for all $\varepsilon \in [1, \bar{\varepsilon}]$. Therefore, either (57) is never binding, or it is binding in an interval of low values of $\varepsilon : [1, \varepsilon_0]$.
11. Statements 7 to 10 imply the following characterization of the optimal path of the control variable:

$$\begin{aligned} q_\varepsilon &= 0 \quad \text{for } \varepsilon \in [1, \varepsilon_0] \text{ if } \varepsilon_0 > 1, \\ (\varepsilon - \gamma_2) U'(q_\varepsilon) &= \gamma_1 C'(q_\varepsilon) \quad \text{for } \varepsilon \in [\varepsilon_0, \hat{\varepsilon}], \text{ and} \\ q_\varepsilon &= \hat{q} \quad \text{for } \varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}] \text{ if } \hat{\varepsilon} < \bar{\varepsilon}; \end{aligned} \quad (68)$$

where

$$\gamma_1 = \frac{\lambda}{2\lambda - 1}, \text{ and } \gamma_2 = \frac{(\lambda - 1)\bar{\varepsilon} - \Sigma_{\hat{\varepsilon}}\varphi^{-1}}{2\lambda - 1}. \quad (69)$$

The two real numbers ε_0 and $\hat{\varepsilon}$ obey: $1 \leq \varepsilon_0 \leq \hat{\varepsilon} \leq \bar{\varepsilon}$.

12. If $\hat{\varepsilon} = \bar{\varepsilon}$ (condition (56) is never binding), then $\Sigma_{\hat{\varepsilon}} = 0$. If $\hat{\varepsilon} < \bar{\varepsilon}$, the first order condition (65) can be simplified using (68) and (69) for $\hat{\varepsilon}$, to obtain

$$\varsigma_{\varepsilon}\varepsilon = (2\lambda - 1)\varphi(\varepsilon - \hat{\varepsilon}) + \Sigma_{\varepsilon} - \Sigma_{\hat{\varepsilon}}. \quad (70)$$

Since $\varsigma_{\varepsilon} = -\dot{\Sigma}_{\varepsilon}$, (70) is a differential equation. Its general solution is:

$$\varsigma_{\varepsilon} = \frac{1}{2}(2\lambda - 1)\varphi + \frac{K}{\varepsilon^2}, \text{ and} \quad (71)$$

$$\Sigma_{\varepsilon} = \Sigma_{\hat{\varepsilon}} - \frac{1}{2}(2\lambda - 1)\varphi(\varepsilon - 2\hat{\varepsilon}) + \frac{K}{\varepsilon}. \quad (72)$$

The constant of integration K can be determined using the condition $\varsigma_{\hat{\varepsilon}} = 0$, so

$$K = -\frac{1}{2}(2\lambda - 1)\varphi\hat{\varepsilon}^2. \quad (73)$$

Also, the definition (64) implies $\Sigma_{\bar{\varepsilon}} = 0$. Therefore,

$$\Sigma_{\hat{\varepsilon}} = \frac{\varphi\bar{\varepsilon}}{2}(2\lambda - 1) \left[1 - 2\frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}}\right)^2 \right]. \quad (74)$$

Combining (74) and (69), we obtain:

$$\gamma_1 + \frac{\gamma_2}{\bar{\varepsilon}} = \frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \frac{1}{2} \left[1 - \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}}\right)^2 \right]. \quad (75)$$

13. Conditional on ε_0 and $\hat{\varepsilon}$, the set of equations (68), (69), and (74) characterize the optimal path of the control variable $\{q_{\varepsilon}\}_{\varepsilon=1}^{\bar{\varepsilon}}$. The optimal path $\{v_{\varepsilon}\}_{\varepsilon=1}^{\bar{\varepsilon}}$ is obtained from (55) and (58). If interior, the optimal values of ε_0 and $\hat{\varepsilon}$ are obtained combining the interior first order condition (66) with the constraints (57) and (56) respectively. The values of ε_0 and $\hat{\varepsilon}$ are at a corner solution if at $\varepsilon_0 = 1$ and/or $\hat{\varepsilon} = \bar{\varepsilon}$ the constraints (57) and (56) are satisfied together with the associated Kuhn-Tucker complementary conditions.

14. The values λ , m , and v_1 solve the following program:

$$\max_{\{\lambda, m, v_1\}} J(\lambda, m, v_1) - im \quad (76)$$

subject to (42).

15. Since λ is the Lagrange multiplier associated with constraint (42). The first order interior conditions of program (76) can be written as follows:

$$i = J_m(\lambda, m, v_1), \text{ and} \quad (77)$$

$$J_{v_1}(\lambda, m, v_1) = 0; \quad (78)$$

together with the constraint (42).

16. Using the Envelope Theorem, (59), (64), and $\varphi = (\bar{\varepsilon} - 1)^{-1}$, conditions (77) and (78) are transformed into:

$$i = \Sigma_{\bar{\varepsilon}} \quad (79)$$

$$1 - \lambda + \Sigma_{\bar{\varepsilon}} = 0. \quad (80)$$

Therefore,

$$\lambda = 1 + i. \quad (81)$$

Conditions (79) and (81) combined with (67) implies that

$$\gamma_1 = \frac{1+i}{1+2i}, \text{ and } \gamma_2 = \frac{i}{1+2i}. \quad (82)$$

17. Define q_1^* to be the solution to $U'(q_1^*) = C'(q_1^*)$. The assumptions about U and C imply $q_1^* > 0$. Substituting (82) into (68) implies that $q_\varepsilon \geq q_1 = q_1^* > 0$. Therefore, constraint (57) is never binding, that is $\varepsilon_0 = 1$.

18. In conclusion, the optimal path $\{q_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}$ is characterized by (68), (75), (82), and $\varepsilon_0 = 1$. For i sufficiently small, this solution satisfies the assumptions made at the head of Statement 1 because of the following reasons. Equation (81) implies $\lambda > 1/2$. For $i = 0$, (79) implies $\Sigma_{\bar{\varepsilon}} = 0$, so constraint (56) is never binding. Continuity implies that for i sufficiently small $m > z_1 > -v_1$.

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