Optimal Stabilization Policies in a Model with Financial Intermediation

Aleksander Berentsen
University of Basel

Christopher J. Waller
University of Notre Dame

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Abstract

We construct a model where money is essential to examine the role of financial intermediaries in the propagation of aggregate shocks. We then derive the optimal stabilization response by the central bank to these shocks where the central bank's objective is to maximize the welfare of the representative agent subject to a seigniorage constraint. We show that financial intermediation can lead to higher consumption volatility and that the optimal policy typically involves providing an elastic supply of currency to smooth nominal interest rates.

1 Introduction

Optimal monetary policy has a long and short-run component. The long-run component focuses on what the optimal trend inflation rate or money growth rate should be. The short-run component is concerned with the optimal stabilization response to economic shocks. The 'science of monetary policy' requires constructing macroeconomic models to study these issues. What attributes should such a model have? First, since the issue is optimal monetary policy, it seems obvious that money should be essential for trade. Second, the model should be based on micro-foundations with optimizing agents. Third, the monetary authority should choose policies that maximize the welfare of the representative agent subject to the constraints of the economic environment.

How well does the current 'science of monetary policy' satisfy these three criteria? New Keynesian (NK) models come close to satisfying the second and third criteria. On the other hand, the NK model fails miserably in terms of satisfying criteria one since they do not explicitly derive the underlying frictions that give rise to money. Instead money is forced into the model, which gives rise to the second drawback of the model. For monetary policy to have real effects there must be nominal price/wage rigidity but this rigidity is simply assumed to exist and is not derived from first principles. Thus, by having poor microfoundations for money in the models, ad hoc assumptions about price and wage rigidity must be used for money to have real effects. A more appealing approach is to model explicitly the frictions that give rise to money. These frictions alone should be sufficient to give monetary policy real effects thus avoiding the need for assumptions on price and wage rigidity.

In this paper we study optimal monetary policy in a dynamic stochastic general equilibrium model in which money is essential. The basic framework is the Berentsen, Camera and Waller (2004) model of money and credit that builds on the standard Lagos-Wright (2004) model of money. We introduce a variety of economic shocks to this economy and examine the optimal stabilization response of the monetary authority for a given long run inflation rate target. The existence of the credit sector generates a nominal interest rate that the monetary authority is able to manipulate via changes in the aggregate money stock. The monetary authority's objective is to maximize the lifetime expected utility of the representative agent subject to being an equilibrium policy and satisfying an exogenous seigniorage requirement. In the absence of any seigniorage requirement, we show that the optimal long run policy is the Friedman rule (a zero nominal interest rate) and stabilization policy is not needed. However, with binding seigniorage requirements that force the monetary authority away from the Friedman rule, stabilization policy has real, welfare improving effects. We show that stabilizing nominal interest rates is welfare improving for some shocks but not others.

The paper proceeds as follows. In Section 2 we describe the environment. In Section 3 the agents optimization problems are presented. Section 4 examines the equilibrium of the economy and the impact of each shock on the allocation. Section 5 contains the optimal stabilization policies of the central bank. Section 6 concludes.

2 The Environment

Time is discrete and in each period there are two markets that open sequentially and are perfectly competitive.[1] There is a $[0,1]$ continuum of infinitely-lived agents and one perishable good produced and consumed by all agents.

At the beginning of the first market agents get a preference shock such that they either can consume or produce. With probability n an agent can consume but cannot produce while with probability $1-n$ the agent can produce but cannot consume. We refer to consumers as buyers and producers as sellers. Agents get utility $\varepsilon u(q)$ from $q > 0$ consumption in the first market, where ε is a preference parameter and $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. Furthermore, we impose that the elasticity of utility $e\,(q) = \frac{qu'(q)}{u(q)}$ is bounded. Producers incur utility cost $c\,(q) = q/a$ from producing q units of output where a is a measure of productivity. To motivate a role for fiat money, we assume that all goods trades are anonymous. In particular, trading histories of agents are private information. Consequently, sellers require immediate compensation so buyers pay with money. There is also no public communication of individual trading outcomes (public memory), which eliminates the use of trigger strategies to support gift-giving equilibria.

The parameters n, a and ε are stochastic. The random variable n has support $[\underline{n}, \overline{n}] \in (0,1)$, a has support $[\underline{a}, \overline{a}]$, $\infty > \overline{a} > \underline{a} > 0$, and ε has support $[\underline{\varepsilon}, \overline{\varepsilon}]$, $\infty > \overline{\varepsilon} > \underline{\varepsilon} > 0$. Let $\omega = (n, a, \varepsilon) \in \Omega$ be the aggregate state in market 1, where $\Omega = [\underline{n}, \overline{n}] \times [\underline{a}, \overline{a}] \times [\underline{\varepsilon}, \overline{\varepsilon}]$ is a closed and compact subset on $\boldsymbol{R}^3_+$. Let $f\,(\omega)$ denote the density function of ω. The shocks follow first order stationary Markov processes with the stationary transition matrix Π. The stationary conditional density function is denoted by $f\,(\omega|\omega_{-1})$. Denote the unconditional expectations $E\,(n) = \hat{n}$, $E(1/a) = 1$

[1] The basic environment is that of Berentsen, Camera and Waller (2004) which is a combination of Lagos and Wright (2003) and Aruoba, Waller and Wright (2003).

and $E(\varepsilon) = 1$. Since shocks to n affect the ratio of sellers to buyers it can be interpreted as an extensive margin aggregate demand shock with high values of n corresponding to many buyers and high demand for goods.[2] Aggregate shocks to ε are intensive margin aggregate demand shocks. Shocks to a are aggregate productivity shocks. If the shocks are independent, then $f(\omega|\omega_{-1}) = f(n, n_{-1}) f(a, a_{-1}) f(\varepsilon, \varepsilon_{-1})$.

In the second market all agents consume and produce, getting utility $U(x)$ from x consumption, with $U'(x) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) \leq 0$.[3] Agents can produce one unit of x with one unit of labor h. Production of x units of output generates disutility h. The discount factor across dates is $\beta \in (0, 1)$. There is no uncertainty in market 2. Adding uncertainty to this market is not interesting because monetary policy will have no effects on real variables in this market since it is a completely frictionless market.

We assume a central bank exists that controls the supply of fiat currency. The growth rate of the money stock is given by $M_t = \gamma M_{t-1}$ where $\gamma > 0$ and M_t denotes the per capita money stock in market 2 in period t. For notational ease variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1. Agents receive lump sum transfers $\tau M_{-1} = (\gamma - 1)M_{-1}$ over the period t. If $\tau < 0$, we assume the central bank has the authority to levy taxes in the form of currency to extract cash from the economy. Some of the transfer is received at the beginning of market 1 and some during market 2. Let $\tau_1 M_{-1}$ and $\tau_2 M_{-1}$ denote the transfers in market 1 and 2 in state $\omega \in \Omega$ respectively with $(\tau_1 + \tau_2) M_{-1} = \tau M_{-1}$. Thus, since there is randomness in the economy, the central bank may choose to make the timing of the injections contingent on the aggregate state of the economy. Note that the growth rate of the money supply is still deterministic.[4] If the timing of the transfers is not state contingent, we say that policy is passive. Otherwise it is active.

As in BCW (2004) there are banks that have a record-keeping technology over financial trans-

[2] When the measures of buyers and sellers are random the aggregate demand for goods is also random and changes on the extensive margin. Although the measure of sellers is random, with linear production costs equilibrium output is entirely demand determined. By varying the number of sellers we also change the amount of idle cash balances in the economy. When banks intermediate deposits, this implies deposits and loans will be random. This suggests that banks may amplify shocks to the economy. We show below that although banks increase consumption in all states they also create more consumption volatility for individual agents.

[3] Following Lagos-Wright (2003), the difference in preferences over the good sold in the last market allows us to impose technical conditions such that the distribution of money holdings is degenerate at the beginning of a period.

[4] Lucas (1990) employs a similar process for the money supply to so that changes in nominal interest rates result purely from allocative shocks and not changes in expected inflation. The same holds true in our model.

actions that allows them to take deposits and make loans. In the first market the banking sector opens before trading and agents can borrow and deposit after observing the shocks. Then, buyers and sellers trade. In the second market the goods market and the banking sector opens simultaneously. We assume net settlements, i.e. all financial claims are settled at the end of the period. This essentially means that loans and deposits cannot be rolled over. Consequently, all financial contracts are one-period contracts. In all models with credit repayment is a serious issue. However, since we focus on stabilization issues, we assume repayment can be enforced on all loans.

Banks accept nominal deposits and pay the nominal interest rate i_d and make nominal loans at nominal rate i. The banking sector is perfectly competitive, so banks take these rates as given. There are no operating costs so the zero profit condition implies that $i_d = i$.

It is straightforward to show that in a symmetric equilibrium in state ω all borrowers take out the same size loan, $l = l(\omega)$, and depositors deposit the same amount $d = d(\omega)$ so that

$$l = \frac{(1-n)\,\mu}{n} d \tag{1}$$

where μ is an indicator variable that equals 1 if a banking system exists and 0 otherwise. We introduce μ in order to compare the allocations when banks exist and when they do not.

The precise sequence of action after agents observe the shocks is as follows. First, the monetary injection $\tau_1 M_{-1}$ occurs. Second, sellers deposit their excess cash and buyers borrow money from the banking sector. Thus, monetary policy can influence market interest rates without doing open market operations – it simply injects money and lets the private sector reallocate the cash thereby causing interest rates to change. Finally, agents move on to the goods market and trade. In the second market the injection $\tau_2 M_{-1}$ occurs and the goods market and banking sector open where all financial claims are settled.

3 Agents' Choices and Value functions

In period t, let $\phi = \phi(\omega)$ be the real price of money in the second market given state ω occurred in market 1. We study equilibria where end-of-period real money balances are time-invariant

$$\phi M = \phi_{-1} M_{-1} = z. \tag{2}$$

We refer to it as a stationary equilibrium.

Consider a stationary equilibrium. Let $V(m_1, \omega_{-1})$ denote the expected value from trading in market 1 with m_1 money balances conditional on the aggregate shock ω_{-1}. Let $W(m_2, l, d)$ denote the expected value from entering the second market with m_2 units of money, l loans, and d deposits when the aggregate state is ω. Note that all quantities and prices are functions of the aggregate state ω, i.e., $m_2 = m_2(\omega)$, $l = l(\omega)$, and $d = d(\omega)$. We suppress this dependence for notational simplicity. In what follows, we look at a representative period t and work backwards from the second to the first market to examine the agents' choices.

3.1 The second market

In the second market agents trade good x and adjust their money balances taking into account cash payments or receipts from the bank. Loans are repaid by borrowers and bank redeem deposits. If an agent has borrowed l units of money, then he pays $(1+i)\,l$ units of money. If he has deposited d units of money, he receives $(1+i)\,d$. The representative agent's program is

$$W(m_2, l, d) = \max_{x,h,m_{1,+1}} \left[U(x) - h + \beta V(m_{1,+1}, \omega)\right] \tag{3}$$

$$\text{s.t.} \quad x + \phi m_{1,+1} = h + \phi\,(m_2 + \tau_2 M_{-1}) + \phi\,(1+i)\,d - \phi\,(1+i)\,l$$

where $m_{1,+1}$ is the money taken into period $t+1$, l is nominal borrowing, and d is nominal deposits in the first market.

Rewriting the budget constraint in terms of h and substituting into (3) yields

$$\begin{aligned} W(m_2, l, d) &= \phi\left[m_2 + \tau_2 M_{-1} - (1+i)\,l + (1+i)\,d\right] \\ &+ \max_{x,m_{1,+1}} \left[U(x) - x - \phi m_{1,+1} + \beta V(m_{1,+1}, \omega)\right]. \end{aligned}$$

The first-order conditions are $U'(x) = 1$ and

$$-\phi_{-1} + \beta V'(m_1, \omega_{-1}) = 0 \tag{4}$$

where the first-order condition for money has been lagged one period. Thus, $V'(m_1, \omega_{-1})$ is the marginal value of taking an additional unit of money into the first market open in period t, and ϕ_{-1} is the real price of money in the second market of period $t-1$ measured in units of utility.

The envelope conditions are

$$W_m = \phi \tag{5}$$

$$W_d = -W_l = \phi\,(1+i)\,. \tag{6}$$

If no banking system exists then $W_d = W_l = 0$. As in Lagos-Wright (2004) the value function is linear in wealth. The implication is that all agents enter the following period with the same amount of money.

3.2 The first market

Let q_b and q_s respectively denote the quantities consumed by a buyer and produced by a seller trading in market 1. Let p be the nominal price of goods in market 1. It is straightforward to show that buyers will never deposit funds in the bank and sellers will never take out loans. Thus, $l_s = d_b = 0$. In what follows we let l denote loans taken out by buyers and d deposits of sellers. We also drop these arguments in $W(m, l, d)$ where relevant for notational simplicity.

An agent who has m_1 money at the opening of the first market given ω_{-1} has expected lifetime utility

$$V(m_1, \omega_{-1}) = \int_\Omega \{n [\varepsilon u(q_b) + W(m_1 + \tau_1 M_{-1} + l - pq_b, l)] + (1-n)[-q_s/a + W(m_1 + \tau_1 M_{-1} - d + pq_s, d)]\} f(\omega|\omega_{-1}) \, d\omega \tag{7}$$

where pq_b is the amount of money spent as a buyer, and pq_s the money received as a seller.

After the shocks are realized, agents become either a buyer or a seller.

Sellers' decisions It is straightforward to show that it is optimal for sellers to deposit all their money balances if $i > 0$. If $i = 0$, they are indifferent since they earn no money. In what follows we assume that if $i = 0$, then $d = 0$. This assumption has no implication for the equilibrium allocation but simplifies the presentation. Thus, the optimal choice for $d = d(\omega)$ satisfies

$$d \begin{cases} = m_1 + \tau_1 M_{-1} & \text{if} \quad i > 0 \\ = 0 & \text{otherwise} \end{cases}, \ \omega \in \Omega \tag{8}$$

A seller's problem is

$$\max_{q_s} [-q_s/a + W(pq_s, d)]$$

Using (5), the first order conditions reduce to

$$1 = ap\phi, \qquad \omega \in \Omega. \tag{9}$$

Since sellers have linear costs in both markets they are indifferent how much they produce in market 1 given (9) holds. In what follows we assume that all sellers produce the same quantities. Note that sellers cannot deposit receipts of cash obtain from selling output.

Buyers' decisions If an agent is a buyer in the first market, his problem is:

$$\max_{q_b, l} [\varepsilon u(q_b) + W(m_1 + \tau_1 M_{-1} + l - pq_b, l)]$$

$$\text{s.t.} \quad pq_b \leq m_1 + \tau_1 M_{-1} + l$$

Notice that buyers can spend more cash than what they bring into the first market since they can borrow cash to supplement their money holdings at the cost of the nominal interest rate. Using (5) the buyer's first-order conditions can be written as

$$u'(q_b) - p\phi - p\lambda = 0, \qquad \omega \in \Omega \tag{10}$$

$$-i\phi + \lambda = 0, \qquad \omega \in \Omega \tag{11}$$

$$\lambda [m_1 + \tau_1 M_{-1} + l - pq_b] = 0, \qquad \omega \in \Omega \tag{12}$$

where $\lambda = \lambda(\omega)$ are the multipliers of the buyer's cash constraints for all states $\omega \in \Omega$.

Define the set of states where the constraint is nonbinding as $\Omega_0 = \{\omega \in \Omega \,|\, \lambda(\omega) = 0\}$. Accordingly, define $\Omega_1 = \{\omega \in \Omega \,|\, \lambda(\omega) > 0\}$. Then, if the constraint is not binding, by using (5), (10) reduces to

$$a\varepsilon u'(q_b) = 1, \qquad \omega \in \Omega_0. \tag{13}$$

Hence trades are efficient.[5] In what follows define the solution of (13) as $q^* = q^*(\omega)$. From (11) it is evident that if $\lambda = 0$ the interest rate in a monetary equilibrium satisfies $i = 0$.

If the constraint is binding, then equations (9), (10) and (11) imply

$$a\varepsilon u'(q_b) = 1 + i, \qquad \omega \in \Omega_1. \tag{14}$$

Since from (11) $i > 0$, we have $a\varepsilon u'(q_b) > 1$ which means trades are inefficient. The buyer spends all of his money, $pq_b = m_1 + \tau_1 M_{-1} + l$, and consumes $q_b = \frac{m_1 + \tau_1 M_{-1} + l}{p}$.

Marginal value of money The marginal value of money satisfies

$$V'(m_1, \omega_{-1}) = \int_\Omega \left\{ n\varepsilon u'(q_b)/p + (1-n)\phi(1+\mu i) \right\} f(\omega|\omega_{-1})\, d\omega$$

In the appendix we show that the value function is concave in m.

Use (9) to replace $p = 1/(a\phi)$ to get

$$V'(m_1, \omega_{-1}) = \int_\Omega \phi \left\{ na\varepsilon u'(q_b) + (1-n)(1+\mu i) \right\} f(\omega|\omega_{-1})\, d\omega. \tag{15}$$

Note that banks increase the marginal value of money because sellers can deposit idle cash and earn interest as opposed to the non-bank case. This is captured by the second term on the right-hand side. In the non-bank case ($\mu = 0$) this term is just $1 - n$.

[5] With n buyers and $1-n$ sellers, the planner maximizes $nu(q_b) - (1-n)c(q_s)$ s.t. $nq_b = (1-n)q_s$. Use the constraint to replace q_s in the maximand. The first-order condition for q_b is $u'(q_b) = c'(q_s)$.

3.3 Equilibrium

We now derive the symmetric monetary steady-state equilibrium. Market clearing implies

$$q_b = \frac{1-n}{n} q_s, \qquad \omega \in \Omega. \tag{16}$$

Note that by symmetry we impose that all sellers produce the same quantity q_s even though they are indifferent regarding how much to produce when (9) holds.

Use (14) to eliminate i and (4) to eliminate $V'(m_1, \omega_{-1})$ from (15) to get

$$\phi_{-1}(\omega_{-1}) = \beta \int_\Omega \phi(\omega) \left\{1 + [n + (1-n)\mu] \left[a\varepsilon u'(q_b) - 1\right]\right\} f(\omega|\omega_{-1})\, d\omega.$$

Then, by using (13) we can rewrite the previous equation as follows:

$$\phi_{-1}(\omega_{-1}) = \beta \int_\Omega \phi(\omega) f(\omega|\omega_{-1})\, d\omega + \beta \int_{\Omega_1} \phi(\omega) [n + (1-n)\mu] \left[a\varepsilon u'(av\theta z(\omega)) - 1\right] f(\omega|\omega_{-1})\, d\omega.$$

Here we have taken into account that in a steady-equilibrium $m_1 = M_{-1}$ so buyer's money holdings satisfy

$$(1+\tau_1)\left(1 - \mu + \frac{\mu}{n}\right) M_{-1} = v\theta M$$

where $\theta = 1 - \mu + \mu/n$ and $v = (1+\tau_1)/(1+\tau)$.[6] Consequently, we have $q_b = \phi a v\theta M = av\theta z(\omega)$ if $\omega \in \Omega_1$, where $z = z(\omega)$ is the real stock of money in state ω.

Finally, multiply the first-order condition by M to get

$$\gamma z_{-1}(\omega_{-1}) = \beta \int_\Omega z(\omega) f(\omega|\omega_{-1})\, d\omega + \beta \int_{\Omega_1} z(\omega) [n + (1-n)\mu] \left[a\varepsilon u'(av\theta z(\omega)) - 1\right] f(\omega|\omega_{-1})\, d\omega.$$

Since we study equilibria where the end-of-period real money balances are time invariant we can write this equation as follows

$$\frac{\gamma}{\beta} z(\omega_{-1}) = \int_\Omega z(\omega) f(\omega|\omega_{-1})\, d\omega + \int_{\Omega_1} z(\omega) [n + (1-n)\mu] \left[a\varepsilon u'(av\theta z(\omega)) - 1\right] f(\omega|\omega_{-1})\, d\omega. \tag{17}$$

Definition 1 *A symmetric monetary steady-state equilibrium is a function $z(\omega)$ that satisfies (17).*

(17) is a functional equation in $z(.)$. If the states are not serially correlated, i.e. if they are i.i.d, then the right-hand side of (17) is independent of ω_{-1}. Consequently, the real stock of money is a constant. If the states are serially correlated, the real stock of money is a function of the state.

We now analyze the optimal response to these shocks when the shocks are serially correlated and when they are not. We begin with the second case.

[6] Note that $\theta = 1$ if there are no banks and $\theta = 1/n$ with banks.

4 Optimal policy without persistence

Before we derive the optimal response to these shocks, we now analyze each shock separately to understand how each one affects the allocation.

When the states are not serially correlated (17) can be written as follows

$$\frac{\gamma-\beta}{\beta} = \int_{\Omega_1} \left[n+(1-n)\mu\right]\left[a\varepsilon u'(av\theta z)-1\right] f(\omega)\, d\omega. \tag{18}$$

4.1 Extensive margin demand shocks

For the analysis of shocks to n, we set $1/a = E(1/a) = 1$ and $\varepsilon = E(\varepsilon) = 1$. It then follows that $\omega = n$. Note that the optimal quantities solve $q^* = u'^{-1}(1)$.

Proposition 1 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*$ for all n at $\gamma = \beta$. For $\gamma > \beta$, the equilibrium is unique with $q = q^*$ if $n \leq \tilde{n}$ and $q < q^*$ if $n > \tilde{n}$, where $\tilde{n} \in (\underline{n}, \bar{n})$. Moreover, $d\tilde{n}/d\gamma < 0$.*

The Friedman rule replicates the first-best allocation. At the Friedman rule agents can self-insure at no cost so there is no role for stabilization policies.[7] Away from the Friedman rule, buyers are constrained when there are many borrowers (high n) but are not constrained when there are many depositors (low n). Since $d\tilde{n}/d\gamma < 0$, the higher is the trend inflation rate, the larger is the range of shocks where the quantity traded is inefficiently low.

We want to compare the equilibrium outcome in the banking economy to the no-banking case when these shocks occur. For this purpose we eliminate for now policy responses to these shocks by setting $\tau_1(n) = \tau$ implying $v(n) = 1$ for all n. We can replicate the no-banking equilibrium simply by setting $\mu = 0$ which implies $\theta = 1$. The quantities consumed in the no-banking equilibrium are $q(n) = z$, where z solves

$$\frac{\gamma-\beta}{\beta} = \int_{\underline{n}}^{\overline{n}} n\left[u'(z)-1\right] dF(n)$$

Hence, in the no banking equilibrium, buyers consume the same quantity across states since they can only spend the cash they bring into market 1 which is independent of the state that is realized.[8] In contrast, when banks exist, idle cash from sellers is deposited and lent back out to buyers. This expands borrowing and leads to higher quantities of goods consumed by buyers. If $n > \tilde{n}$, then

[7] This is very similar to the result found by Ireland (1996). He shows that even with nominal price stickiness, there is no need to stabilize aggregate demand shocks at the Friedman rule.

[8] Note however that aggregate consumption in the economy is increasing in n.

increasing n means there are more buyers and fewer deposits to lend out. This drives up nominal interest rates, which decreases consumption. Thus, individual consumption is high in low demand states and low in high demand states.

Corollary 1 *Without banks $dq/dn = 0$. With banks we have $dq/dn = -q/n \leq 0$ for $n > \tilde{n}$ and $dq/dn = 0$ for $n \leq \tilde{n}$.*

The interesting aspect of this result is that while financial intermediation raises average consumption across states, is also causes interest rates to fluctuate thereby making individual consumption more volatile. To see this note that the nominal interest rate is $i = 0$ for $n \leq \tilde{n}$ and $i = u'(q) - 1 \geq 0$ for $n > \tilde{n}$. Note also that the expected interest rate satisfies[9]

$$\frac{\gamma - \beta}{\beta} = \int_{\tilde{n}}^{\overline{n}} idF(n).$$

Note that the expected nominal interest rate is increasing in γ since $d\tilde{n}/d\gamma < 0$.

4.2 Intensive margin demand shocks

To study ε shocks we set $a = 1$ and $n = \hat{n}$. It then follows that $\omega = \varepsilon$. Note that in this case the optimal quantities solve $q^*(\varepsilon) = u'^{-1}(1/\varepsilon)$.

Proposition 2 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*(\varepsilon)$ for all ε at $\gamma = \beta$. For $\gamma > \beta$ the equilibrium is unique with $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$ and $q = q^*(\varepsilon)$ for $\varepsilon < \tilde{\varepsilon}$.*

Once again the Friedman rule replicates the first-best allocation. At the Friedman rule there is no role for stabilization. Away from the Friedman rule, buyers are constrained in high marginal utility states but not in low states.

Corollary 2 *With a passive monetary policy $dq/d\varepsilon = 0$ for $\varepsilon > \tilde{\varepsilon}$ and $dq/d\varepsilon > 0$ for $\varepsilon \leq \tilde{\varepsilon}$.*

This Corollary follows from the fact that when buyers are constrained $(\varepsilon > \tilde{\varepsilon})$ $q(\varepsilon) = \theta z$ which does not depend on ε. Thus, even though buyers have a high marginal utility for consumption and would like to buy more goods at the prevailing market price, they cannot. The reason is that their holdings of money balances are unchanged and the market price is not a function of ε. Thus, from (14) the excess demand for goods simply leads to an increase in the nominal interest rate on loans

[9] For $n < \tilde{n}$, buyers have more real balances than they need to buy the efficient quantity. However, since $i = 0$ in these states, there is no gain from depositing the excess cash balances in the bank.

such that agents are able to buy the same quantity of goods. For $\varepsilon \leq \tilde{\varepsilon}$, buyers have more than enough real balances to buy the efficient quantity. So when ε increases, they simply spend more of their money balances.

4.3 Aggregate productivity shocks

To study aggregate productivity shocks, we set $\varepsilon = 1$ and $n = \hat{n}$. It then follows that $\omega = a$. Note that in this case the optimal quantities are $q^*(a) = u'^{-1}(1/a)$. Denote $R(q) = -qu''(q)/u'(q)$.

Proposition 3 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*(a)$ for all a at $\gamma = \beta$. For $\gamma > \beta$ the equilibrium is unique. If $R(q) \geq 1$, then $q < q^*(a)$ for $a > \tilde{a}$ and $q = q^*(a)$ for $a < \tilde{a}$. If $R(q) < 1$, then $q = q^*(a)$ for $a \geq \tilde{a}$ and $q < q^*(a)$ for $a < \tilde{a}$.*

As productivity increases, the marginal costs of producing fall which increases the efficient quantity. It also implies that prices fall. The issue is whether prices fall sufficiently far to raise real balances, az, enough to buy the higher efficient quantity. This in turn depends on preferences – for sufficiently concave preferences it is not sufficient, while for preferences sufficiently linear, prices increase more than enough to buy the efficient quantity. Thus, we have the following:

Corollary 3 *With passive policy, $dq/da > 0$ for all a.*

4.4 Optimal Stabilization

What is the optimal response of the central bank to these shocks? We assume the central bank's stabilization policy maximizes the welfare of the representative agent for a given steady-state inflation rate. It does so by choosing the quantities consumed and produced in each state subject to the constraint that the chosen quantities satisfy the conditions of a competitive equilibrium. The policy is implemented by choosing the injections τ_1 and τ_2 accordingly. This is a standard Ramsey problem. The growth rate of the money supply is determined exogenously to finance a fixed amount of government consumption in market 2. This implies $\gamma > 1$.

It is straightforward to show that the expected lifetime utility of the representative agent at the beginning of period t is given by

$$(1-\beta)V(M_{-1}) = U(x) - x - g + \int_{\Omega} n[\varepsilon u(q) - q/a] f(\omega, \omega_{-1}) d\omega$$

where g is the real amount of government spending that is financed in market 2. It is obvious that $x = x^*$ so all that remains is to choose $q = q(\omega)$.

The Ramsey problem facing the central bank is

$$\underset{q}{Max} \int_{\Omega} n\left[\varepsilon u(q) - q/a\right] f(\omega, \omega_{-1})\, d\omega$$

$$\text{s.t. } \frac{\gamma - \beta}{\beta} = \int_{\Omega} \left[n + (1-n)\mu\right]\left[a\varepsilon u'(q) - 1\right] f(\omega, \omega_{-1})\, d\omega \tag{19}$$

$$q \leq q^*$$

where μ is an indicator variable which is equal to 1 when there are no banks and equal to 0 when there are banks.

Sufficient conditions for a maximum are

$$\begin{aligned} n\left[a\varepsilon u'(q) - 1\right] + \lambda\left[n + (1-n)\mu\right] a^2 \varepsilon u''(q) &= 0 \text{ and} \\ a\varepsilon u''(q)\, u''(q) - \left[a\varepsilon u'(q) - 1\right] u'''(q) &> 0 \text{ for all } \omega. \end{aligned} \tag{20}$$

The second order condition is satisfied if either $u'''(q) \leq 0$ or if $u'''(q) > 0$ and marginal utility is log-concave.

The first thing to note is that the planner never chooses $q = q^*(\omega)$ for any state. This is in contrast to the decentralized equilibrium where $q = q^*(\omega)$ for some range of all three shocks when $\gamma > \beta$. The reason is that the planner attempts to smooth consumption across states whereas no state contingent contracts are traded amongst the agents themselves. He does this by setting $i > 0$. Consequently, although $i = 0$ is the optimal policy for all states, unless it can be done for all states, it is optimal to never set $i = 0$. Hence, zero nominal interest rates should be an all-or-nothing policy.

The second thing to note is that while banks do affect the *level* of consumption across states that a planner chooses, the existence of banks does not affect the *ratios* of consumption across states. To see this, simply taking the ratio of (20) for two realizations of the shocks

$$\frac{n_j\left[a_j \varepsilon_j u'\left(q^j\right) - 1\right]}{n_k\left[a_k \varepsilon_k u'\left(q^k\right) - 1\right]} = \frac{a_j^2 \varepsilon_j u''\left(q^j\right)}{a_k^2 \varepsilon_k u''\left(q^k\right)}$$

which is independent of μ. In short, the financial system merely relaxes the constraint for the planner but has no impact on how consumption is allocated across states.

We now want to characterize the planner's choices across states. With respect to extensive margin shocks we state the following:

Proposition 4 *The constrained planner's choice of quantities yields* $\frac{dq}{dn} = 0$ *when there are no banks* $(\mu = 0)$ *and*

$$\frac{dq}{dn} = -\frac{u''(q)\left[a\varepsilon u'(q) - 1\right]}{na\varepsilon u''(q)\, u''(q) + n\left[a\varepsilon u'(q) - 1\right] u'''(q)} \geq 0 \tag{21}$$

with banks ($\mu = 1$).

According to Proposition (4), without banks the planner chooses the same consumption in each state which implies $v^n = v$. The reason is that although the central bank can alter q via its choice of τ_1, doing so would only create consumption variability which is welfare reducing. Consequently, the best stabilization policy is to keep quantities constant across states. Note that this implies aggregate consumption still varies across states.

In contrast, with banks the planners optimal choice of individual consumption is increasing in n. Note also that this just the opposite from what happens when the central bank is passive. With a passive policy, buyers consume more in low n states since loans are plentiful and cheap. However, this is not optimal. From the viewpoint of the representative agent looking into the future, he wants to consume more when he is more likely to desire consumption, which is when he is a buyer (high n). However, these are the states when there is little liquidity in the banking system since the number of depositors is low. Thus, what the planner would like to do is to increase liquidity when aggregate demand is high and deposits are low and reduce it when aggregate demand is low and deposits are high. To accomplish this, the central bank injects more cash into the system in high n states and less cash in low n states. These transfers are chosen to generate the quantities solving (20).

An interesting implication of this policy is that the central bank is essentially providing an elastic supply of currency to the economy – when demand for liquidity is high, the central bank provides additional currency and withdraws it when the demand for liquidity is low. Note what this does NOT imply – a constant nominal interest rate across states, which would imply that consumption is the same in all states.

With respect to intensive margin demand shocks we have:

Proposition 5 *The constrained planner's choice of quantities yields*

$$\frac{dq}{d\varepsilon} = -\frac{u''(q)}{\varepsilon \left\{ a\varepsilon u''(q)\, u''(q) - \left[a\varepsilon u'(q) - 1 \right] u'''(q) \right\}} \geq 0 \tag{22}$$

with or without banks.

Unlike the passive policy equilibrium, where $dq/d\varepsilon = 0$ when $q < q^*(\varepsilon)$, the planner wants consumption to be increasing in ε. As before, when agents desire more consumption, it is optimal to give them more. This is achieved by injecting more liquidity into the system to bring down nominal interest rates thereby allowing agents to buy more goods when it is so desired.[10]

Proposition 6 *The constrained planner's choice of quantities yields*

$$\frac{dq}{da} = -\frac{u''(q)\{\varepsilon a u'(q) + 2[a\varepsilon u'(q) - 1]\}}{a^2 \varepsilon u''(q) u''(q) + a[a\varepsilon u'(q) - 1] u'''(q)} \geq 0 \tag{23}$$

with or without banks.

5 Optimal policy with persistence

For the rest of the paper we focus on shocks to the number of buyers and sellers since banks play a key role in the propagation of these shocks. To analyze these shocks we begin with the case where there are only two states.

5.1 Two states

Consider $n \in \{n^H, n^L\}$ where $n^H > n^L > 0$. Assume that there are banks ($\mu = 1$). Assume that the shocks follow a *stationary finite state first-order Markov process*, which is summarized in the following transition probability matrix:

	n_{t+1}^H	n_{t+1}^L
n_t^H	π^H	$1-\pi^H$
n_t^L	$1-\pi^L$	π^L

where π^j denotes the probability that state j follows state j, $i \neq j$. Accordingly, $1 - \pi^j$ is the probability that state i follows state j. We assume

$$\pi^H, \pi^L \geq 1/2 \tag{24}$$

With strict inequality there is no persistence.

Conjecture a steady state equilibrium with two values for the real stock of money z^H and z^L. We will show below under which conditions such an equilibrium exists. The average inflation is $E\left(\frac{M_{t+1}}{M_t}\right) = \gamma$, while the expected gross real return on money is

$$R^j = E_t^j\left[\frac{\phi_{t+1}}{\phi_t^j}\right] = E_t^j\left[\frac{z_{t+1}}{\gamma z_t^j}\right] = \frac{1}{\gamma}\frac{z^j\pi^j + z^i\left(1-\pi^j\right)}{z^j} \equiv \frac{1}{\xi^j},$$

[10] Recall that the extra liquidity is withdrawn in market two.

which is negatively associated with expected inflation γ. In our analysis we will focus on ξ^j rather than γ since ξ^j is proportional to γ.

The Friedman rule We can write the two first-order conditions (17) as follows

$$\frac{\gamma}{\beta}z^H = z^H u'\left(q^H\right)\pi^H + z^L u'\left(q^L\right)\left(1-\pi^H\right) \tag{25}$$

$$\frac{\gamma}{\beta}z^L = z^H u'\left(q^H\right)\left(1-\pi^L\right) + z^L u'\left(q^L\right)\pi^L \tag{26}$$

Substract from both sides the expected real stock of money and rewrite these equations to get

$$\frac{\gamma}{\beta}z^H - E_t^H\left[z_{t+1}\right] = z^H\pi^H\left[u'\left(q^H\right)-1\right] + z^L\left(1-\pi^H\right)\left[u'\left(q^L\right)-1\right] \tag{27}$$

$$\frac{\gamma}{\beta}z^L - E_t^L\left[z_{t+1}\right] = z^H\left(1-\pi^L\right)\left[u'\left(q^H\right)-1\right] + z^L\pi^L\left[u'\left(q^L\right)-1\right] \tag{28}$$

Evidently, the first-best allocation can be attained with a policy (the Friedman rule) that sets the expected return on money $\frac{1}{\xi^j}$ equal to the real interest rate $\frac{1}{\beta}$, i.e.,

$$\frac{1}{\beta} = \frac{1}{\gamma}\frac{z^j\pi^j + z^i\left(1-\pi^j\right)}{z^j}$$

Note that Friedman rule requires less deflation than in a deterministic model if $\frac{z^j\pi^j + z^i\left(1-\pi^j\right)}{z^j} > 1$. If $\frac{z^j\pi^j + z^i\left(1-\pi^j\right)}{z^j} > 1/\beta$, the Friedman rule requires a positive average rate of inflation. Note that the Friedman rule requires different policies for the two states.

In order to derive the equilibrium away from the Friedman rule we have to distinguish between two ranges for γ. We first derive the equilibrium when the rate of inflation is high as made clear below.

5.1.1 High γ

Assume γ is sufficiently large so that the cash constraint binds in both states.

Proposition 7 *A monetary equilibrium where agents are constrained in both states exists and is unique if*

$$f\left(\bar{z}^L\right) \geq \breve{g}\left(\bar{z}^L\right) \text{ and } g\left(\bar{z}^H\right) \geq \breve{f}\left(\bar{z}^H\right) \tag{29}$$

where the quantities $\breve{g}\left(\bar{z}^L\right)$, $f\left(\bar{z}^L\right)$, $g\left(\bar{z}^H\right)$ *and* $\breve{f}\left(\bar{z}^H\right)$ *are defined in the proof.*

Proof. The first-order conditions (25) and (26) can be expressed as follows

$$z^H = \frac{z^L}{\left(1-\pi^L\right)}\left[\pi^H + \frac{\beta}{\gamma}u'\left(\frac{n^L z^L}{\upsilon^L}\right)\left(1-\pi^L-\pi^H\right)\right] \tag{30}$$

$$z^L = \frac{z^H}{\left(1-\pi^H\right)}\left[\pi^L + \frac{\beta}{\gamma}u'\left(\frac{n^H z^H}{\upsilon^H}\right)\left(1-\pi^L-\pi^H\right)\right]. \tag{31}$$

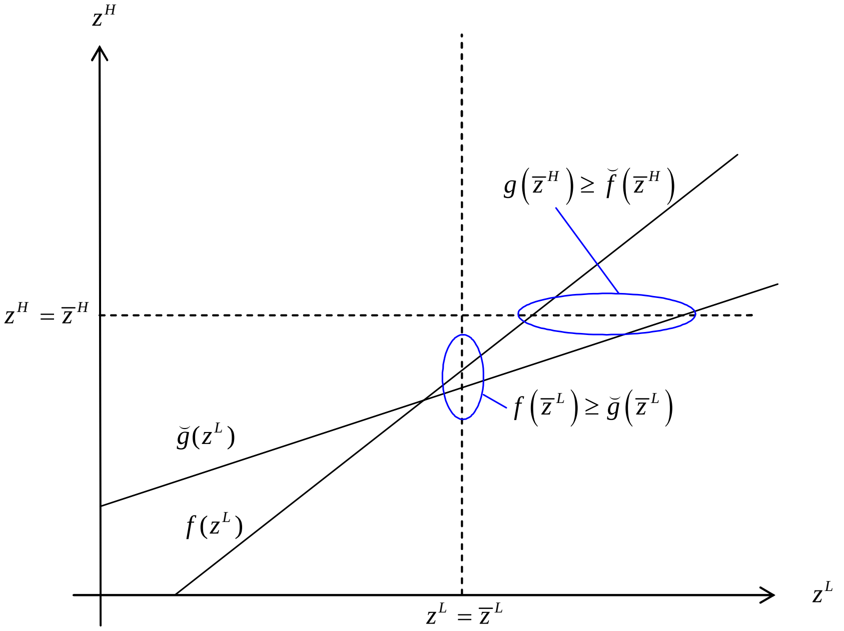
Define the sets

$$Z^L = \left\{ z^L \in R \left| \pi^H + \frac{\beta}{\gamma} u' \left(\frac{n^L z^L}{\upsilon^L} \right) \left(1 - \pi^L - \pi^H\right) \geq 0 \right. \right\} \text{ and}$$
$$Z^H = \left\{ z^H \in R \left| \pi^L + \frac{\beta}{\gamma} u' \left(\frac{n^H z^H}{\upsilon^H} \right) \left(1 - \pi^L - \pi^H\right) \geq 0 \right. \right\}.$$

Equation (30) defines a function $z^H = f\left(z^L\right)$ which is strictly increasing in $z^L \in Z^L$. We can therefore invert it to get $z^L = \breve{f}\left(z^H\right)$. Furthermore if $\pi^H + \frac{\beta}{\gamma} u'\left(\frac{n^L z^L}{\upsilon^L}\right)\left(1 - \pi^L - \pi^H\right) \geq 0$, then $z^H = 0$. Thus $z^H = 0$ for a strictly positive value of z^L since $\left(1 - \pi^L - \pi^H\right) < 0$. Equation (31) defines a function $z^L = g\left(z^H\right)$ which is strictly increasing in $z^H \in Z^H$. We can therefore invert it to get $z^H = \breve{g}\left(z^L\right)$. Furthermore, if $\pi^L + \frac{\beta}{\gamma} u'\left(\frac{n^H z^H}{\upsilon^H}\right)\left(1 - \pi^L - \pi^H\right) = 0$, then $z^L = 0$. Thus $z^L = 0$ for a strictly positive value of z^H since $\left(1 - \pi^L - \pi^H\right) < 0$.

Define $\bar{z}^L = \frac{n^L q^*}{\upsilon^L}$ and $\bar{z}^H = \frac{n^H q^*}{\upsilon^H}$. Then, the properties of these functions imply that a monetary equilibrium exists where agents are constrained in both states if and only if

$$f\left(\bar{z}^L\right) = \frac{\bar{z}^L}{\left(1 - \pi^L\right)} \left[\pi^H + \frac{\beta}{\gamma} u'\left(q^*\right)\left(1 - \pi^L - \pi^H\right) \right] \geq \breve{g}\left(\bar{z}^L\right) \text{ and} \tag{32}$$
$$g\left(\bar{z}^H\right) = \frac{\bar{z}^H}{\left(1 - \pi^H\right)} \left[\pi^L + \frac{\beta}{\gamma} u'\left(q^*\right)\left(1 - \pi^L - \pi^H\right) \right] \geq \breve{f}\left(\bar{z}^H\right) \tag{33}$$



Existence requires $\breve{g}(\bar{z}^L) \leq f(\bar{z}^L)$ and $g(\bar{z}^H) \geq \breve{f}(\bar{z}^H)$.

Since $f\left(z^L\right)$ and $\breve{g}\left(z^L\right)$ are strictly increasing in $z^L \in Z^L$ the equilibrium is unique if it exists. ■

Logarithmic utility For illustration let $u(x) = \ln x$. It then follows that

$$z^j u'\left(q^j\right) = \frac{n^j}{\upsilon^j} \text{ and } q^* = 1$$

which is independent of z^j. From (25) and (26) we get

$$q^H = \frac{\beta}{\gamma}\left[\zeta\left(1-\pi^H\right)+\pi^H\right] \text{ and } q^L = \frac{\beta}{\gamma}\left[\frac{1}{\zeta}\left(1-\pi^L\right)+\pi^L\right]$$

where $\zeta = n^L\upsilon^H/\left(n^H\upsilon^L\right)$.

For this equilibrium to exist we need $q^H, q^L < q^* = 1$. Thus, we must have

$$\zeta\left(1-\pi^H\right)+\pi^H < \frac{\gamma}{\beta} \text{ and } \frac{1}{\zeta}\left(1-\pi^L\right)+\pi^L < \frac{\gamma}{\beta}$$

which is consistent with (32) and (33). Note that the real stock of money requires that

$$z^H = \frac{n^H q^H}{\upsilon^H} = \frac{\beta}{\gamma}\left[\frac{n^L}{\upsilon^L}\left(1-\pi^H\right)+\frac{n^H}{\upsilon^H}\pi^H\right] \text{ and } z^L = \frac{n^L q^L}{\upsilon^L} = \frac{\beta}{\gamma}\left[\frac{n^H}{\upsilon^H}\left(1-\pi^L\right)+\frac{n^L}{\upsilon^L}\pi^L\right]$$

Passive policy and Friedman rule Consider the special case of a passive policy where $\upsilon^H = \upsilon^L$ so $\zeta = \frac{n^L}{n^H} < 1$ and $\pi^L = \pi^H = \pi > 1/2$. In this case we have $q^H < q^L$ and $z^H > z^L$ since $\pi^H + \pi^L \geq 1$. Then if we are in the high n state in period t , the economy is more likely to be in the high demand state again tomorrow. Since agents are more likely to be buyers in period $t+1$, there is an increase in the demand for money in market 2 which increases ϕ^H thereby increasing z^H. If the state is n^L in period t, then agents are more likely to be sellers next period, so the demand for money falls causing z^L to fall.

Consider the Friedman rule. Assume that we are in the low state. Then, it requires that

$$\begin{aligned}
\frac{1}{\beta} &= \frac{1}{\gamma}\frac{z^L\pi^L + z^H\left(1-\pi^L\right)}{z^L} \\
&= \frac{1}{\gamma}\frac{\left[n^H\left(1-\pi^L\right)+n^L\pi^L\right]\pi^L + \left[n^L\left(1-\pi^H\right)+n^H\pi^H\right]\left(1-\pi^L\right)}{\left[n^H\left(1-\pi^L\right)+n^L\pi^L\right]} \\
&= \frac{1}{\gamma}\frac{n^H\left(1-\pi^L\right)\pi^L + n^L\pi^L\pi^L + n^L\left(1-\pi^H\right)\left(1-\pi^L\right) + n^H\pi^H\left(1-\pi^L\right)}{\left[n^H\left(1-\pi^L\right)+n^L\pi^L\right]} \\
&= \frac{1}{\gamma}\frac{2n^H\left(1-\pi\right)\pi + n^L\pi\pi + n^L\left(1-\pi\right)\left(1-\pi\right)}{\left[n^H\left(1-\pi\right)+n^L\pi\right]} \\
&= \frac{1}{\gamma}\frac{2n^H\left(1-\pi\right)\pi + n^L\left(1-2\pi\left(1-\pi\right)\right)}{\left[n^H\left(1-\pi\right)+n^L\pi\right]} \\
&= \frac{1}{\gamma}\frac{n^L + 2\left(n^H-n^L\right)\left(1-\pi\right)\pi}{\left[n^H\left(1-\pi\right)+n^L\pi\right]}
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{n^L + 2\left(n^H - n^L\right)(1-\pi)\pi}{\left[n^H(1-\pi) + n^L\pi\right]} &\leq 1 \\
n^L + 2\left(n^H - n^L\right)(1-\pi)\pi &\leq \left[n^H(1-\pi) + n^L\pi\right] \\
\text{since } \left(n^H - n^L\right)\left[2(1-\pi)\pi - 1 + \pi\right] &\leq 0
\end{aligned}$$

Assume that we are in the high state. Then, it requires that

$$\begin{aligned}
\frac{1}{\beta} &= \frac{1}{\gamma}\frac{z^H\pi^H + z^L\left(1-\pi^H\right)}{z^H} \\
&= \frac{1}{\gamma}\frac{\left[n^L(1-\pi) + n^H\pi\right]\pi + \left[n^H(1-\pi) + n^L\pi\right](1-\pi)}{\left[n^L(1-\pi) + n^H\pi\right]} \\
&= \frac{1}{\gamma}\frac{2n^L(1-\pi)\pi + n^H\pi\pi + n^H(1-2\pi(1-\pi))}{\left[n^L(1-\pi) + n^H\pi\right]} \\
&= \frac{1}{\gamma}\frac{n^H - 2\left(n^H - n^L\right)(1-\pi)\pi}{\left[n^L(1-\pi) + n^H\pi\right]}
\end{aligned}$$

Consumption smoothing Note that the central bank can make $q^L = q^H$ by setting

$$\frac{n^H}{n^L} = \frac{\upsilon^H}{\upsilon^L}$$

This stabilizes consumption across states and requires a procyclical policy when n is high υ is high. The question then is whether or not this is the optimal policy – would the planner choose consumption smoothing? The answer is almost certainly no. The planner wants more consumption in the high state and less in the low state just as before.

5.1.2 Low γ

Now suppose that agents are only constrained in state H or only constrained in state L.

Proposition 8 *A monetary equilibrium where agents are constrained in state H and unconstrained in state L exists and is unique if*

$$\breve{g}\left(\bar{z}^L\right) \geq f\left(\bar{z}^L\right) \text{ and } g\left(\bar{z}^H\right) \geq \breve{f}\left(\bar{z}^H\right) \tag{34}$$

where the quantities $\breve{g}\left(\bar{z}^L\right)$, $f\left(\bar{z}^L\right)$, $g\left(\bar{z}^H\right)$ and $\breve{f}\left(\bar{z}^H\right)$ are defined in the proof.

A monetary equilibrium where agents are constrained in state L and unconstrained in state H exists and is unique if

$$\breve{g}\left(\bar{z}^L\right) \leq f\left(\bar{z}^L\right) \text{ and } g\left(\bar{z}^H\right) \leq \breve{f}\left(\bar{z}^H\right). \tag{35}$$

Proof. If agents are only constrained in state H we have

$$q^H = \frac{\upsilon^H z^H}{n^H} < q^* \leq \frac{\upsilon^L z^L}{n^L}$$

Accordingly, the first-order conditions (25) and (26) can be expressed as follows

$$\begin{aligned}
\frac{\gamma}{\beta} z^H &= z^H u'\left(\frac{\upsilon^H z^H}{n^H}\right)\pi^H + z^L u'(q^*)\left(1-\pi^H\right) \\
\frac{\gamma}{\beta} z^L &= z^H u'\left(\frac{\upsilon^H z^H}{n^H}\right)\left(1-\pi^L\right) + z^L u'(q^*)\pi^L
\end{aligned}$$

Rewrite these equations to get

$$z^H\left(1-\pi^L\right) = z^L\left[\pi^H + \frac{\beta}{\gamma}u'(q^*)\left(1-\pi^H-\pi^L\right)\right] \tag{36}$$

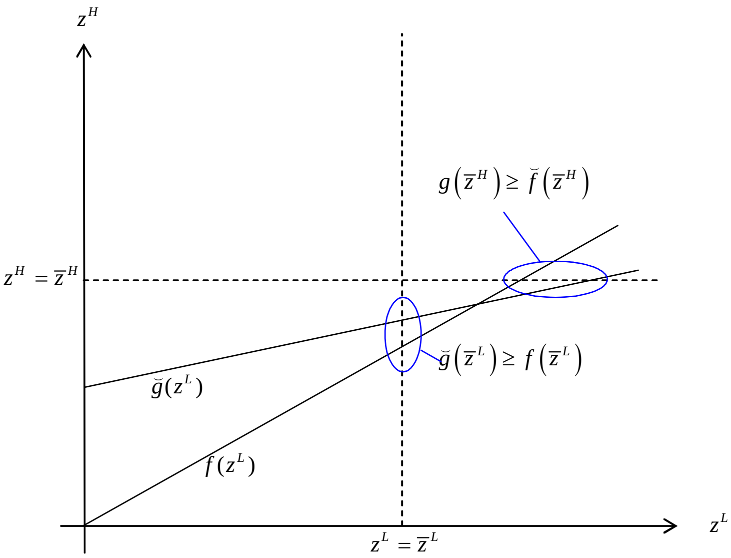
$$z^L\left(1-\pi^H\right) = z^H\left[\pi^L + \frac{\beta}{\gamma}u'\left(\frac{\upsilon^H z^H}{n^H}\right)\left(1-\pi^L-\pi^H\right)\right] \tag{37}$$

Equation (36) defines a function $z^H = f\left(z^L\right)$ which is strictly increasing in z^L. We can therefore invert it to get $z^L = \breve{f}\left(z^H\right)$. Furthermore we have $z^H = 0$ if $q^L = 0$. Equation (37) defines a function $z^L = g\left(z^H\right)$ which is strictly increasing in $z^H \in Z^H$. We can therefore invert it to get $z^H = \breve{g}\left(z^L\right)$. Furthermore, if $\pi^L + \frac{\beta}{\gamma}u'\left(q^H\right)\left(1-\pi^L-\pi^H\right) = 0$, we have $z^L = 0$. Thus $z^L = 0$ for a strictly positive value of z^H since $\left(1-\pi^L-\pi^H\right) < 0$.

Thus existence requires that

$$\breve{g}\left(\bar{z}^L\right) \geq f\left(\bar{z}^L\right) = \frac{\bar{z}^L}{\left(1-\pi^L\right)}\left[\pi^H + \frac{\beta}{\gamma}u'(q^*)\left(1-\pi^H-\pi^L\right)\right] \text{ and} \tag{38}$$

$$g\left(\bar{z}^H\right) = \frac{\bar{z}^H}{\zeta\left(1-\pi^H\right)}\left[\pi^L + \frac{\beta}{\gamma}u'(q^*)\left(1-\pi^L-\pi^H\right)\right] \geq \breve{f}\left(\bar{z}^H\right) \tag{39}$$



Existence requires $\breve{g}(\bar{z}^L) \geq f(\bar{z}^L)$ and $g(\bar{z}^H) \geq \breve{f}(\bar{z}^H)$.

Since $f\left(z^L\right)$ and $\breve{g}\left(z^L\right)$ are strictly increasing in z^L the equilibrium is unique if it exists.

The proof when agents are constrained in state L only is similar and therefore omitted. ■

Logarithmic utility Assume agents are constrained in state H only. With log utility we have

$$q^H = \frac{\upsilon^H z^H}{n^H} \leq q^* = 1 < \frac{\upsilon^L z^L}{n^L}$$

From (25) and (26) we get

$$z^L = \frac{\beta n^H \left(1-\pi^L\right)}{\upsilon^H \left(\gamma - \pi^L \beta\right)} \text{ and } z^H = \frac{\beta n^H \left[\pi^H \gamma + \beta\left(1-\pi^H-\pi^L\right)\right]}{\gamma \upsilon^H \left(\gamma - \beta\pi^L\right)} \tag{40}$$

For this equilibrium to exist we need

$$\frac{\upsilon^H z^H}{n^H} \leq 1 \Rightarrow \frac{\left[\pi^H\gamma + \beta\left(1-\pi^H-\pi^L\right)\right]}{\left(\gamma - \beta\pi^L\right)} \leq \frac{\gamma}{\beta} \Rightarrow \beta\left(\pi^H + \pi^L - 1\right) \leq \gamma$$

which holds for all values of π^H and π^L if $\gamma \geq \beta$.

We also need

$$1 < \frac{\upsilon^L z^L}{n^L} \Rightarrow 1 < \zeta \frac{\beta\left(1-\pi^L\right)}{\left(\gamma - \pi^L\beta\right)} \Rightarrow \frac{\gamma}{\beta} < \zeta\left(1-\pi^L\right) + \pi^L$$

Thus, for $\gamma \in \left(\beta, \zeta\left(1-\pi^L\right)+\pi^L\right]$, (40) is an equilibrium.

5.2 k states

Now suppose there are $k > 2$ shocks taking the value $n^j \in [\underline{n}, \overline{n}]$ with $n^{j+1} > n^j$ for all j. Continue to assume that we have log utility. The discrete number of shocks version of (17)

$$\frac{\gamma}{\beta} z^s = \sum_{j=1}^{k} z^j u'\left(\frac{\upsilon^j z^j}{n^j}\right)\pi^{sj}$$

Conjecture that for all $n^j \geq \tilde{n}$, the agents are constrained while for $n^j < \tilde{n}$ they are unconstrained. We thus have

$$z^s = \frac{\beta}{\gamma}\sum_{j=\tilde{n}}^{k} z^j \pi^{sj} + \frac{\beta}{\gamma}\sum_{j=1}^{\tilde{n}-1}\frac{n^j}{\upsilon^j}\pi^{sj} \quad \text{for } s = 1, 2, .., k$$

Ordering these equations from the lowest value state to the highest value state gives us a system of equations that can be written as

$$AZ = B$$

where Z is a k x 1 matrix whose i^{th} element is z^i, B is a k x 1 matrix whose i^{th} element is

$$c^i = -\frac{\beta}{\gamma}\sum_{j=1}^{\tilde{n}-1}\frac{n^j}{\upsilon^j}\pi^{ij}$$

and A is a k x k matrix that can be partioned as

$$A = \begin{bmatrix} I & P \\ \varnothing & T \end{bmatrix}$$

where I is a $\tilde{n}-1$ x $\tilde{n}-1$ identity matrix, $\varnothing$ is a $k+1-\tilde{n}$ x $\tilde{n}-1$ matrix of zeros and P is a $\tilde{n}-1$ x $k+1-\tilde{n}$ matrix whose ij^{th} element is $\frac{\beta}{\gamma}\pi^{ij}$. Finally, T is a $k+1-\tilde{n}$ x $k+1-\tilde{n}$ matrix whose off diagonal elements are $\frac{\beta}{\gamma}\pi^{ij}$ and whose diagonal elements are $\left(\frac{\beta\pi^{ii}-\gamma}{\beta}\right)$. Thus a steady state is the solution to

$$Z = A^{-1}B$$

Conditions need to be imposed that each z is positive. It then simply comes down to deriving conditions such that

$$\frac{\upsilon^i z^i}{n^i} \leq q^* = 1 < \frac{\upsilon^j z^j}{n^j}$$

for $i < \tilde{n}$ and $j \geq \tilde{n}$. This should give us conditions on the range of γ.

6 Conclusion

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Appendix

Marginal value of money Differentiating (7) with respect to m_1

$$V'(m_1,\omega_{-1}) = \int_{\Omega} n\left[\varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} + W_m\left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) + W_l\tfrac{\partial l}{\partial m_1}\right] + (1-n)\left[-(1/a)\tfrac{\partial q_s}{\partial m_1} + W_m\left(1 + p\tfrac{\partial q_s}{\partial m_1} - \tfrac{\partial d}{\partial m_1}\right) + W_d\tfrac{\partial d}{\partial m_1}\right] f(\omega|\omega_{-1})\,d\omega$$

Recall from (5) and (6) that $W_m = \phi$ and $W_d = -W_l = \phi(1+i_d)$ $\forall m_2$. Furthermore, $\frac{\partial q_s}{\partial m_1} = 0$ because the quantity a seller produces is independent of his money holdings. We also know that $\frac{\partial d_s}{\partial m_1} = 1$ since a seller deposits all his cash when $i > 0$. Hence,

$$V'(m_1,\omega_{-1}) = \int_{\Omega}\left\{n\left[\varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} + \phi\left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) - \phi(1+i)\tfrac{\partial l}{\partial m_1}\right] + (1-n)\phi\left[(1+(1-\mu)i)\right]\right\} f(\omega|\omega_{-1})\,d\omega$$

Since $i > 0$ implies $pq_b = m_1 + l$ we have $\left(1 - p\frac{\partial q_b}{\partial m_1} + \frac{\partial l}{\partial m_1}\right) = 0$. Furthermore, note that Note that

$$\varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} - \phi(1+i)\tfrac{\partial l}{\partial m_1} = \varepsilon u'(q_b)\tfrac{\partial q_b}{\partial m_1} - \phi(1+i)\left[p\tfrac{\partial q_b}{\partial m_1} - 1\right]$$
$$= \tfrac{\partial q_b}{\partial m_1}\left[\varepsilon u'(q_b) - \phi(1+i)p\right] + \phi(1+i) = \phi(1+i) = \phi a\varepsilon u'(q_b).$$

Hence

$$V'(m_1,\omega_{-1}) = \int_{\Omega}\phi\left\{na\varepsilon u'(q_b) + (1-n)(1+(1-\mu)i)\right\} f(\omega|\omega_{-1})\,d\omega$$

Define $m^* = pq^*$ where $q^* = u'^{-1}(1/(a\varepsilon))$. Then if $m_1 < m^*$, $0 < q_b < q^*$, implying $\frac{\partial q_b}{\partial m_1} > 0$ so that $V''(m_1,\omega_{-1}) < 0$. If $m_1 \geq m^*$, $q_b = q^*$ implying $\frac{\partial q_b}{\partial m_1} = 0$, so that $V''(m_1,\omega_{-1}) = 0$. Thus, $V(m_1,\omega_{-1})$ is concave $\forall m$.■

Proof of Proposition 1. The proof involves two steps. We first show that for a given z a critical value $\tilde{n}$ exist such that if $n < \tilde{n}$ trades are efficient and if $n > \tilde{n}$ they are inefficient. We then show that an unique z exist.

Step 1: Critical value $\tilde{n}$. In a steady-equilibrium $m_1 = M_{-1}$ so buyer's money holdings satisfy

$$(1+\tau_1)\left(1 - \mu + \frac{\mu}{n}\right)M_{-1} = \upsilon\theta M \tag{41}$$

where $\theta = 1 - \mu + \mu/n$ and $\upsilon = (1+\tau_1)/(1+\tau)$. Note that $\theta = 1$ if there are no banks and $\theta = 1/n$ with banks. Thus, in any equilibrium $pq \leq \upsilon\theta M$. Then, the seller's first-order condition $p = 1/\phi$ implies that in any equilibrium $q \leq \upsilon\theta z$. The efficient quantity is defined to be $u'(q^*) = 1$. If trades are efficient, $q = q^* \leq \upsilon\theta z$ otherwise $q = \upsilon\theta z < q^*$. Consequently, there is a critical value $\tilde{n}$ for which

$$\tilde{n} = \tilde{n}(z) = \frac{\mu\upsilon z}{q^* - (1-\mu)\upsilon z} \tag{42}$$

Note that $\frac{\partial \tilde{n}}{\partial z} \geq 0$. This implies that $q = q^*$ for $n \leq \tilde{n}$ and $q = \upsilon\theta z$ for $n > \tilde{n}$.

Step 2: Existence and uniqueness. From (18) the first-order condition for $m_{1,+1}$ satisfies

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{n}}^{\overline{n}} [n + (1-n)\mu] \left[u'(q) - 1\right] f(n)\, dn.$$

where $q = q_b(n)$.

From (42) and the fact that $q = \upsilon\theta z < q^*$ if $n > \tilde{n}$, we can write it as

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{n}}^{\overline{n}} [n + (1-n)\mu] \left[u'(\upsilon\theta z) - 1\right] f(n)\, dn \equiv RHS.$$

The right-hand side is a function of z only. For $z \to 0$ we have

$$\lim_{z\to 0} RHS = \lim_{z \to 0} \int_{\underline{n}}^{\overline{n}} [n + (1-n)\mu] \left[u'(\upsilon\theta z) - 1\right] f(n)\, dn = \infty.$$

For $z = q^*/\upsilon\theta$ we have $\tilde{n} = \overline{n}$ and therefore

$$RHS\Big|_{z=\frac{q^*}{\upsilon\theta}} = 0 \leq \frac{\gamma - \beta}{\beta}.$$

Hence a monetary equilibrium exists.

To establish uniqueness use Leibnitz rule to find

$$\frac{\partial RHS}{\partial z} = \upsilon\theta \int_{\tilde{n}}^{\overline{n}} [n + (1-n)\mu] u''(\upsilon\theta z) f(n)\, dn - u'(q^*) f(\tilde{n}) \frac{\partial \tilde{n}}{\partial z} \leq 0$$

So the the right-hand side is monotonically decreasing in z. Consequently, we have a unique z, denoted z, such that

$$\tilde{n} = \frac{\mu\upsilon z}{q^* - (1-\mu)\upsilon z}$$

and

$$q = q^* \text{ if } n \leq \tilde{n} \text{ and } q < q^* \text{ otherwise.}$$

Note that if $\mu = 0$, $\tilde{n} = 0$ so the inefficient quantity is consumed for all n for $\gamma > \beta$.

Finally, recall that, due to idiosyncratic trade shocks and financial transactions, money holdings are heterogeneous after the first market closes. Therefore, if we set $m_1 = M_{-1}$, the money holdings of agents at the opening of the second market are $m_2 = 0$ for buyers and $m_2 = \left(\frac{n}{1-n}\right)\theta\upsilon M$ for sellers. Solving for equilibrium consumption and production in the second market, with $x^* = U'^{-1}(1)$,gives

Trading history:	Production in the last market:
Buy	$h_b = x^* + \frac{q}{a\theta} + \left(\frac{\theta-1}{\theta}\right) e(q)\, u(q)$
Sell	$h_s = x^* - \left(\frac{n}{1-n}\right) \left[\frac{q}{a\theta} + \left(\frac{\theta-1}{\theta}\right) e(q)\, u(q)\right]$

Given our range of n, $\left(\frac{n}{1-n}\right)$ is bounded. Since we assumed that the elasticity of utility $e(q)$ is bounded, we can scale $U(x)$ such that there is a value $x^* = U'^{-1}(1)$ greater than the last term for all $q \in [0, q^*]$. Hence, h_s is positive for for all $q \in [0, q^*]$ ensuring that the equilibrium exists. ■

Proof that banks lower aggregate consumption variability:

TO DO

Proof of Proposition 2: In a steady state equilibrium the real value of money is constant so $\phi M = \phi_{-1} M_{-1} = z$. Thus, using (**??**) and (14) we can write (**??**) as

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\varepsilon u'(q) - 1\right] dF(\varepsilon) \tag{43}$$

In any equilibrium $pq \leq \upsilon\theta M$. Then, the seller's first-order condition $p = 1/\phi$ implies

$$q \leq \upsilon\theta z$$

The efficient quantity is $q^*(\varepsilon) = u'^{-1}(1/\varepsilon)$. Thus, if trades are efficient we have $q = q^*(\varepsilon) \leq \upsilon\theta z$ which requires that

$$\frac{q^*(\varepsilon)}{\upsilon\theta} \leq z \tag{44}$$

Consequently, since - as we show below - z is strictly decreasing in γ, for a given distribution of shocks, there exists a critical value for ε, denotes $\tilde{\varepsilon}$ such that if $\varepsilon \leq \tilde{\varepsilon}$ $q = q^*(\varepsilon)$ and if $\varepsilon > \tilde{\varepsilon}$ then $q < q^*(\varepsilon)$. The critical value satisfies

$$\tilde{\varepsilon} = \frac{1}{u'(\upsilon\theta z)}$$

Note that $\tilde{\varepsilon}$ is decreasing in z and attains zero at $z = 0$.

Using this expression we can obtain a single expression in z

$$\frac{\gamma - \beta}{\beta} = \int_{\tilde{\varepsilon}}^{\overline{\varepsilon}} \left[\varepsilon u'(\upsilon\theta z) - 1\right] dF(\varepsilon)$$

The right-hand side expression is monotonically decreasing in z. To see this note that

$$\frac{\partial RHS}{\partial z} = f(\tilde{\varepsilon})\, \tilde{\varepsilon}^2 u''(q)\, \upsilon\theta + \int_{\tilde{\varepsilon}}^{\overline{\varepsilon}} \left[\varepsilon u''(q)\, \upsilon\theta\right] dF(\varepsilon) < 0$$

Thus there exists a unique value of z such that ....■

Proof of Proposition 3: Taking the expectation of (15) with respect to the marginal utility and using (4) yields

$$\frac{\phi_{-1}}{\phi\beta} = \int_{\underline{a}}^{\overline{a}} \left[n a u'(q) + (1-n)(1+i)\right] dF(a) \tag{45}$$

where $q_b^{\varepsilon} = \left(\frac{1-n}{n}\right) q_s^{\varepsilon} = q$.

In a steady state equilibrium the real value of money is constant so $\phi M = \phi_{-1} M_{-1} = z$. Thus, using (**??**) and (14) we can write (45) as

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{a}}^{\overline{a}} \left[a u' (q) - 1 \right] dF (a) \tag{46}$$

In any equilibrium $pq \leq \upsilon \theta M$. Then, the seller's first-order condition $p = \frac{1}{a\phi}$ implies

$$q \leq a \upsilon \theta z$$

The efficient quantity is $q^* (a) = u'^{-1} (1/a)$. Thus, if trades are efficient we have $q = q^* (a) \leq a \upsilon \theta z$ which requires that

$$\frac{q}{a \upsilon \theta} \leq z$$

There are two cases. Assume first that $R (q) = -q u'' (q) / u' (q) \geq 1$ and that for a given distribution of shocks, there exists a critical value for a, denotes $\tilde{a}$ such that if $a \leq \tilde{a}$ $q = q^* (a)$ and if $a > \tilde{a}$ then $q < q^* (a)$. The critical value satisfies

$$\tilde{a} u' (\tilde{a} \upsilon \theta z) = 1.$$

Then $R (q) = -q u'' (q) / u' (q) \geq 1$ implies that $\frac{d\tilde{a}}{dz} \geq 0$. Using this expression we can obtain a single expression in z

$$\frac{\gamma - \beta}{\beta} = \int_{\tilde{a}}^{\overline{a}} \left[a u' (a \upsilon \theta z) - 1 \right] dF (a)$$

The right-hand side is monotonically decreasing in z. To see this note that

$$\frac{\partial RHS}{\partial z} = -f (\tilde{a}) \tilde{a}^2 u' (\tilde{a} \upsilon \theta z) \frac{d\tilde{a}}{dz} + \int_{\tilde{a}}^{\overline{a}} \left[a u'' (q) a \upsilon \theta \right] dF (a) < 0$$

Consequently, there exists a unique value of z denoted by z such that $\tilde{a}$ solves $\tilde{a} u' (\tilde{a} \upsilon \theta z) = 1$ so that if $a \leq \tilde{a}$ $q = q^* (\varepsilon)$ and if $a > \tilde{a}$ then $q < q^* (a)$.

Assume next that $R (q) = -q u'' (q) / u' (q) < 1$ and that for a given distribution of shocks, there exists a critical value for a, denotes $\tilde{a}$ such that if $a < \tilde{a}$ $q < q^* (\varepsilon)$ and if $a \geq \tilde{a}$ then $q = q^* (a)$. The critical value satisfies

$$\tilde{a} u' (\tilde{a} \upsilon \theta z) = 1.$$

Then $R (q) = -q u'' (q) / u' (q) < 1$ implies that $\frac{d\tilde{a}}{dz} < 0$. Using this expression we can obtain a single expression in z

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{a}}^{\tilde{a}} \left[a u' (a \upsilon \theta z) - 1 \right] dF (a)$$

The right-hand side is monotonically increasing in z. To see this note that

$$\frac{\partial RHS}{\partial z} = f\left(\tilde{a}\right)\tilde{a}^2 u'\left(\tilde{a}v\theta z\right)\frac{d\tilde{a}}{dz} + \int_{\underline{a}}^{\tilde{a}} \left[a u''\left(q\right) a v\theta\right] dF\left(a\right) < 0$$

Consequently, there exists a unique value of z denoted by z such that $\tilde{a}$ solves $\tilde{a}u'\left(\tilde{a}v\theta z\right) = 1$ if $a < \tilde{a}$ $q < q^*\left(a\right)$ and if $a \geq \tilde{a}$ then $q = q^*\left(a\right)$.■

Proof of Proposition 4: ■

Proof of Proposition 6: ■

Proof of Proposition 5: ■