# Wage Distribution with a Two-sided Job Auction

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**Abstract**

We derive a wage distribution in a model where homogenous unemployed workers and homogenous vacancies can send or receive wage offers. We solve the mixed strategies for wage offers and for the fractions of vacancies and unemployed engaged in sending or receiving offers. The aggregate density function for wages is low and increasing at small wages, low and decreasing at high wages, and high and u-shaped at middle-range wages. An increase in the relative supply of labour decreases the average wage. The mixed strategy equilibrium is evolutionarily stable and utilitywise equivalent to auction where the number of competitors is known.

Key words: wage distribution, job search, auctions

JEL codes: J64, J31, J41, D44

## 1 Introduction

Labour markets constitute such a large part of the economy, and for many people leisure is the only significant endowment, that issues of labour economics can rarely be mean-

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ingfully addressed in partial equilibrium models. There are roughly two types of general equilibrium models used in modelling labour markets. One is a search model pioneered by Mortensen and Pissarides (see e.g. Pissarides, 2000) where unemployed and firms are involved in a time consuming activity of looking for each other. In these models the meetings are pairwise, and wages are determined using the Nash bargaining solution or some analogous procedure. The central concept is so called matching function which tells how fast the parties find each other.

In the other branch of models the meetings of the unemployed and vacancies are governed by an urn-ball matching where, say, the employers are contacted by the workers (see e.g. Montgomery, 1991). The advantage of this approach is that the matching function can be determined endogenously, and that it makes multiple meetings and detailed wage formation possible.

There are several empirical findings that are hard to come by in theoretical models. One of these is the empirical wage distribution. A typical wage distribution for observationally identical workers is hump-shaped and right skewed (DiNardo, Fortin and Lemieux, 1996). Wage distributions have been generated by search models with varying results. In Burdett and Mortensen (1998), workers receive wage offers from employers at an exogenous rate, and workers search also on the job. With identical workers and identical firms, the wage offer density function is increasing. The wage offer density can be declining only if offers are made by firms that differ in productivity, or if workers are heterogenous in productivity.

There are articles (Mortensen, 2000; Bontemps, Robin and van den Berg, 2000) that generate distributions of wages that more closely resemble the observed ones. To achieve this, one needs heterogeneity of workers or firms, and some special features of the matching function. These features are not derivable from the basics of the model but are just assumed. Mortensen (2000) considers an on-the-job search model where workers receive wage offers from firms that can make match-specific investments after the firm and worker have met. The firms are heterogenous ex post with respect to the amount of capital. Firms who offer higher wages invest more in match-specific capital, because workers with higher wages have a lower probability to quit. The resulting wage offer density can be increasing, decreasing, or hump-shaped. However, for the density to be hump-shaped, it is required that the production function is of Cobb-Douglas type, and that the parameter of the production function and the exogenous reservation wage fall within certain limits. Bontemps, Robin and van den Berg (2000) allow firms to have different technologies, and they show that a suitable distribution of employer productivity can lead to a hump-shaped distribution for wages.

In this article we demonstrate that one can, with very simple economic reasoning, generate a significant improvement to the wage distribution when one uses an urn-ball model, even though firms and workers are homogenous. We construct a model where unemployed workers can send wage demands to vacant firms that hire the worker who has demanded the lowest wage. Simultaneously, vacant firms can send wage offers to unemployed workers who accept the highest offer. It becomes clear that one cannot use the ideas of this article if one sticks to the search models. We get three kind of results in this article. First, we generate a wage distribution that is first increasing, in the end decreasing, and u-shaped in the middle. Thus, we do not get exactly the wage distribution observed empirically but one that still has several desirable features. Our model also predicts that an increase in the relative supply of labour decreases the average wage. Second, we show that there are three possible equilibria in our model, and of these three only one features an interesting wage distribution. But this is the unique evolutionarily stable equilibrium. Most of the literature has focused on one of the non-stable equilibria. We assume that when several, say, workers contact an employer they do not know how many workers happen to contact that particular employer. This means that when workers make their wage demands, they must use a mixed strategy in equilibrium. We derive the mixed strategy explicitly. Our third result is that in utility terms the mixed strategy is equivalent to a mechanism where the workers know the number of their competitors, and wages are determined in an auction. Kultti (1999) shows the equivalence between such auctions and posted prices; we thus know that all three mechanisms are equivalent in utility terms.

We describe the general idea of the model in Section 2. Sections 3-6 consider a static model that is sufficient to generate a wage distribution. We solve the wage demands and offers in Section 3. In Section 4 we solve the equilibrium fractions of unemployed and vacancies who send offers and who receive them. Section 5 analyses the evolutionary stability of the equilibrium. Section 6 presents the main result of this article, the distribution of realised wages. Section 7 presents the idea of two dynamic versions of the model and their main results. In the Appendix we derive most of the results of Sections 3-6 as well as the analyses of the dynamic models. Section 8 concludes.

# 2 The Model

In the most general setting we consider, everything is in the model, i.e. it is a true general equilibrium model. There are $W$ workers and $E$ employers. Some of them are matched with each other in productive activities, while others are looking for a partner. The number, or measure, of unemployed is denoted by $u$ and the number of vacancies by $v$. Procuction happens in pairs, therefore

$$W - u = E - v. \tag{1}$$

Time is discrete and extends to infinity, and all agents discount future at the common discount factor $\delta < 1$. A matched pair produces output worth a unit each period, and a worker who is employed at wage $w$ gets the wage each period as long as the employment relationship lasts, and correspondingly the employer gets $1 - w$ each period. Utilities are linear such that a worker's utility is $w$, and firm's utility is $1 - w$. Unmatched agents get zero utility.

We focus on the market in a steady state, and for this we need that the matches dissolve every once and a while. We just assume an exogenous separation probability $b$. Each period a match dissolves with probability $b$, and the firm and the worker enter the pool of vacancies and unemployed. This separation probability is not just something we need to do steady state analysis but it is a real feature of real labour markets, and it makes possible to study the duration of unemployment, though not an issue in this article.

One of the crucial features of our analysis is that we determine the equilibrium market structure. Usually it is assumed that unemployed contact vacancies or vice versa. We do not know which is the better assumption, and consequently we allow for both possibilities and determine which case emerges in equilibrium. To this end we postulate that there are two submarkets. Fraction $x \in [0, 1]$ of unemployed workers and fraction $y \in [0, 1]$ of vacancies are in the 'vacancy market' where unemployed workers contact vacancies. Each job seeker, who decides to go to that market, chooses randomly one of the $yv$ vacancies and sends an application accompanied with a wage demand.

The other crucial feature is that the workers do not know which firms the other workers apply to, nor do they know their wage demands. This is an auction with identical valuations but unknown number of bidders. Each firm that has received at least one application hires the worker who has asked the lowest wage. We could say that in the vacancy maket, vacancies stay (or wait), and workers move.

In the 'job seeker market', each of the $(1 - y)\, v$ vacancies sends a wage offer to one of the $(1 - x)\, u$ unemployed workers. The vacancies do not know how much the other vacancies offer and to whom. A job seeker then chooses the firm that has offered the highest wage. We solve the equilibrium fractions $x$ and $y$ as functions of $u/v$, the distributions of wage offers and demands, and realised wages.

We focus on symmetric strategies regarding wage demands and offers and probabilities of going to either market. It is clear that there are no pure strategy equilibria, or equilibria with a mass point for that matter, as to wage demands and offers. The heuristic reason is easy to understand by assuming that there is a pure strategy equilibrium where, say, the unemployed demand wage $w$. There is a positive probability that a particular vacancy is contacted by more than one unemployed. When this happens the probability of a job seeker of getting the job is at most one half. Making a wage demand slightly less than $w$ is a profitable deviation, as then the deviator gets the job for certain, i.e. there is a discrete increase in his probability of getting a job while the wage remains practically the same.[1]

The possibility of two markets is important theoretically because it allows us to deter-

[1] For a formal argument see Kultti and Virrankoski (2003) where in an analogous setting it is shown that there exist only non-atomic mixed strategy equilibria in symmetric strategies. Moreover, it is shown that the support of the mixed strategy must be an interval.

mine the market structure endogenously. It turns out that whether unemployed contact vacancies, or vice versa, or whether two markets with these features exist simultaneously depends on the ratio of unemployed to vacancies. The distribution of wages depends on who contacts whom, and if it is just assumed that contacs take place one way or the other, there is a chance that the wrong, i.e. non-equilibrium, modelling decision is made. This then produces an incorrect wage distribution.

The aim of this article is to derive the wage distribution produced by the urn-ball models, where there are possibly two markets, and where wage offers and demands originate from a mixed strategy. For this purpose it is sufficient to study a static model where the only things of importance are the measures of unemployed and vacancies. This model is got by setting the discount factor to zero and the separation rate to unity, so that each match lasts exactly one period. A slightly more general model is got by assuming that the discount factor is strictly positive but the separation rate is still unity. This corresponds to the dynamic model where it is usually assumed that those who match exit the market and are replaced by identical agents. Here this is one possible interpretation but if one wants to think also this as a special case of the general model it must be assumed that the agents do not remember with whom they have been matched in the previous periods. We conduct most of the analysis via the static model but in the appendix we provide the full equilibrium analysis of the two dynamic models, too.

# 3 Distributions of Wage Demands and Offers

We assume that there are two submarkets; in one market vacancies contact unemployed workers by sending them wage offers, whereas in the other market unemployed workers send wage demands to vacancies. We examine these two case separately and start from the first one.

## 3.1 Vacancies Send Offers to Unemployed Workers

In this market each vacancy sends a wage offer to one randomly chosen unemployed worker. The Poisson parameter that governs the arrivals of offers to workers in this market is $\phi$, which is the ratio of the number of offer-sending vacancies to the the number of offer-receiving workers. (If all unemployed and vacancies are in this market, then $\phi = v/u$.) Let $V_s$ be the utility of a vacancy that sends an offer $w$. Vacancies use a mixed strategy with cumulative distribution function $H(w)$ with support $[b, B]$. The utility of a vacancy is

$$\begin{aligned} V_s &= e^{-\phi}(1-w) + \phi e^{-\phi}\left(1-(w)\right)H(w) + ... + \frac{\phi^k e^{-\phi}}{k!}\left(1-w\right)\left(H(w)\right)^k + ... \qquad (2) \\ &= (1-w)e^{-\phi(1-H(w))}. \end{aligned}$$

In the first term on the right-hand side, $e^{-\phi}$ is the probability that the worker to whom the vacancy sends an offer does not get an offer from any other vacancy, and the vacancy gets profit $1-w$. In the second term, $\phi e^{-\phi}$ is the probability that the worker gets an offer from one other vacancy, and the vacancy we look at manages to hire the worker if the other vacancy's offer is lower than $w$, this happens with probability $H(w)$. The rest of the terms capture the probability that the worker gets offers from exactly $k$ other vacancies, times the probability that they offer less than $w$. The mixed strategy gives $(1-b)e^{-\phi(1-H(b))} = (1-B)e^{-\phi(1-H(B))}$. That is, a vacancy's utility is the same from offer $b$ as from offer $B$. The lowest offer $b$ equals zero, and using $b = 0$, $H(b) = 0$ and $H(B) = 1$, the upper limit of the wage offers is $B = 1 - e^{-\phi}$. The utility of a vacancy is therefore

$$V_s = e^{-\phi}. \qquad (3)$$

Using $V_s = (1-w)e^{-\phi(1-H(w))} = e^{-\phi}$ we get

$$H(w) = -\frac{1}{\phi}\ln(1-w) \qquad (4)$$

with support $w \in \left[0, 1-e^{-\phi}\right]$. The density function is

$$h\left(w\right) \equiv H'(w) = \frac{1}{\phi(1-w)}, \qquad (5)$$

which is increasing in $w$.

In a market where vacancies send offers, the utility of a vacancy is $V_s = e^{-\phi}$. The expected utility of an unemployed worker in this market is equal to $U_r$:

$$
\begin{aligned}
U_r &= \int_b^B \sum_{k=1}^{\infty} \frac{\phi^k e^{-\phi}}{k!} h(w) k \left(H(w)\right)^{k-1} w dw \qquad (6) \\
&= \int_b^B \sum_{k=1}^{\infty} \frac{\phi^k e^{-\phi}}{k!} \frac{1}{\phi\left(1-w\right)} k \left(\frac{\ln\left(1-w\right)^{-1}}{\phi}\right)^{k-1} w dw \\
&= \int_b^B \sum_{k=1}^{\infty} \frac{\left(\ln\left(1-w\right)^{-1}\right)^{k-1}}{(k-1)!} e^{-\phi} \frac{w}{1-w} dw \\
&= \int_b^B \sum_{k=0}^{\infty} \frac{\left(\ln\left(1-w\right)^{-1}\right)^{k}}{k!} e^{-\phi} \frac{w}{1-w} dw = \int_b^B e^{-\phi} e^{\ln(1-w)^{-1}} \frac{w}{1-w} dw \\
&= e^{-\phi} \int_b^B \frac{w}{\left(1-w\right)^2} dw = e^{-\phi} \int_b^B \left(\frac{1}{\left(1-w\right)^2} - \frac{1}{1-w}\right) dw \\
&= e^{-\phi} \left[\ln\left(1-B\right) + \frac{1}{1-B} - \ln\left(1-b\right) - \frac{1}{1-b}\right] \\
&= 1 - e^{-\phi} - \phi e^{-\phi}.
\end{aligned}
$$

The probability that the job seeker gets $k$ offers is $\frac{\phi^k e^{-\phi}}{k!}$. The probability of getting offer $w$ is $h(w)$, and the probability that the wage offered by vacancy $i$ is the highest is $\left(H(w)\right)^{k-1}$: all $k-1$ offers must be lower than $w$. The highest offer can be made by any of the $k$ vacancies. In the second line we use 4 and 5 .

### 3.2 Unemployed Workers Send Applications to Vacancies

Let $U_s$ be the utility of a job seeker who sends an application with wage demand $w$. Unemployed use a mixed strategy with cumulative distribution function $F(w)$ with support $[a, A]$. Let $\theta$ the appropriate Poisson parameter that governs the meeting probabilities (If all workers and firms are in this market, then $\theta = u/v$.) We have

$$
\begin{aligned}
U_s &= e^{-\theta} w + \theta e^{-\theta} \left(1 - F(w)\right) w + ... + \frac{\theta^k e^{-\theta}}{k!} \left(1 - F(w)\right)^k w + ... \qquad (7) \\
&= w e^{-\theta} \left[1 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \left(1 - F(w)\right)^k\right] = w e^{-\theta} \sum_{k=0}^{\infty} \frac{\theta^k \left(1 - F(w)\right)^k}{k!} \\
&= w e^{-\theta F(w)}.
\end{aligned}
$$

In the above, $e^{-\theta}$ is the probability that the vacancy to whom the worker sends an offer does not get an offer from any other worker, thus the worker get $w$. In the rest of the terms, $\frac{\theta^k e^{-\theta}}{k!}$ is the probability that the vacancy gets applications from $k$ other workers. In these cases, $(1 - F(w))^k$ is the probability that all these wage demands are higher than $w$, thus the vacancy rejects these applications and hires our worker.

The utility of an unemployed is the same for all $w \in [a, A]$, especially $ae^{-\theta F(a)} = Ae^{-\theta F(A)}$. Clearly, $A = 1$ because the probability that the unemployed in question is the only applicant is positive. Then $ae^{-\theta F(a)} = e^{-\theta} \Rightarrow a = e^{-\theta}$, and

$$U_s = e^{-\theta}. \tag{8}$$

Next we solve $F(w)$. We have $U_s = e^{-\theta} = we^{-\theta F(w)}$. Taking logarithms results in $\ln e^{-\theta F(w)} = \ln e^{-\theta} - \ln w \Leftrightarrow \theta F(w) = \theta + \ln w$, and the resulting distribution function is

$$F(w) = 1 + \frac{\ln w}{\theta}, \tag{9}$$

with support $w \in \left[e^{-\theta}, 1\right]$. The density function is

$$f(w) \equiv F'(w) = \frac{1}{\theta w}, \tag{10}$$

which is decreasing in $w$.

In the market where unemployed workers send applications, the utility of an unemployed is $U_s = e^{-\theta}$. A vacancy receives $k$ applications, and the probability that the wage asked by unemployed worker $i$ is the lowest is $(1 - F(w))^{k-1}$. The expected utility of a vacancy in this market is equal to $V_r$:

$$
\begin{aligned}
V_r &= \int_a^A \sum_{k=0}^{\infty} \frac{\theta^k e^{-\theta}}{k!} f(w) k \left(1 - F(w)\right)^{k-1} (1-w) dw \qquad (11) \\
&= \sum_{k=1}^{\infty} \frac{\theta^k e^{-\theta}}{k!} \left[ - \left(1 - F(A)\right)^k + \left(1 - F(a)\right)^k \right] \\
&\quad - \int_a^A \sum_{k=1}^{\infty} \frac{\theta^k e^{-\theta}}{k!} f(w) k \left(1 - F(w)\right)^{k-1} w dw \\
&= 1 - e^{-\theta} - \int_a^A \sum_{k=1}^{\infty} \frac{\theta^k e^{-\theta}}{k!} \frac{1}{\theta w} k \left( \frac{\ln w^{-1}}{\theta} \right)^{k-1} w dw \\
&= 1 - e^{-\theta} - \int_a^A \sum_{k=0}^{\infty} \frac{\left(\ln w^{-1}\right)^k}{k!} e^{-\theta} dw = 1 - e^{-\theta} - \int_a^A \frac{e^{-\theta}}{w} dw \\
&= 1 - e^{-\theta} - e^{-\theta} \left( \ln 1 - \ln e^{-\theta} \right) \\
&= 1 - e^{-\theta} - \theta e^{-\theta}.
\end{aligned}
$$

The probability that the vacancy gets $k$ applications is $\frac{\theta^k e^{-\theta}}{k!}$. The probability of receiving wage demand $w$ is $f(w)$, and the probability that the wage demanded by worker $i$ is the lowest is $\left(1 - F(w)\right)^{k-1}$: all other $k-1$ demands must be higher. The lowest demand can be made by any of the $k$ job seekers.

# 4 Equilibrium Market Structure

In an equilibrium where two markets co-exist, the utility of a vacancy that sends offers is the same as the utility of a vacancy who receives wage demands from unemployed workers. The same equivalence condition between sending offers and receiving them holds for unemployed workers, too. That is, we have $V_s = V_r$ and $U_s = U_r$, and inserting the utilities derived above yields

$$
\begin{aligned}
e^{-\phi} &= 1 - e^{-\theta} - \theta e^{-\theta}, \qquad (12) \\
e^{-\theta} &= 1 - e^{-\phi} - \phi e^{-\phi}. \qquad (13)
\end{aligned}
$$

Equation (12) is called vacancies' equilibrium condition $VE$, and equation (13) is called unemployed worker's equilibrium condition $UE$.

**Proposition 1** *The mixed strategies in wage offers are utilitywise equivalent to an auction where the bidders know the number of competitors.*

The utilities $V_r = 1 - e^{-\theta} - \theta e^{-\theta}$ and $U_s = e^{-\theta}$ given above are the same the utilities for a seller and buyer in the static version of Kultti (1999). In his model buyers contact sellers randomly just like the job seekers contact the vacancies in the present model, except that the buyers do not send price offers but engage in a Bertrand competition after it has been revealed how many buyers arrived in a seller's location.

When both conditions hold, we have $\theta e^{-\theta} = \phi e^{-\phi}$, and after substitution we get

$$\phi = \frac{\theta e^{-\theta}}{1 - e^{-\theta} - \theta e^{-\theta}}, \tag{14}$$

$$\theta = \frac{\phi e^{-\phi}}{1 - e^{-\phi} - \phi e^{-\phi}}. \tag{15}$$

Using (14) and (15) in $\theta e^{-\theta} = \phi e^{-\phi}$ results in

$$1 - e^{-\theta} - \theta e^{-\theta} - e^{\frac{-\theta e^{-\theta}}{1 - e^{-\theta} - \theta e^{-\theta}}} = 0, \tag{16}$$

$$1 - e^{-\phi} - \phi e^{-\phi} - e^{\frac{-\phi e^{-\phi}}{1 - e^{-\phi} - \phi e^{-\phi}}} = 0. \tag{17}$$

The solution of (16) and (17) is $\theta = \phi \approx 1.146$ which is denoted by $\theta_0$. This means that if both markets co-exist, the Poisson parameter that governs the arrival rates is the same, $\theta_0$, in both markets. Denoting $u/v$ by $\alpha$, the equilibrium fractions of vacancies and unemployed workers in the two markets satisfy

$$\theta_0 = \frac{\alpha x}{y} = \frac{1-y}{\alpha(1-x)}, \tag{18}$$

and after a few steps we have

$$x = \frac{\theta_0(\alpha\theta_0 - 1)}{\alpha\left(\theta_0^2 - 1\right)}, \tag{19}$$

$$y = \frac{\alpha\theta_0 - 1}{\theta_0^2 - 1}. \tag{20}$$

**Proposition 2** *The vacancy market and the job-seeker market co-exist and $x$ and $y$ are unique if $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$.*

**Proof.** The two markets co-exist only if $x \in (0,1)$ and $y \in (0,1)$. By 19 and 20 this holds only if $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$. The uniqueness is directly seen. ■

The two markets co-exist only if there are roughly equally many vacancies and unemployed workers in the economy. Because $x = \frac{\theta_0}{\alpha} y$ and $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$, we get $\frac{\theta_0}{\alpha} \in (1, 1.313)$, which implies that $x > y$ in equilibrium. If $\alpha \notin \left(\frac{1}{\theta_0}, \theta_0\right)$, search is one-sided. We can directly use a result derived in Kultti, Miettunen, Takalo, and Virrankoski (2004)[2]:

**Proposition 3** *i) If* $\alpha < \frac{1}{\theta_0}$*, then* $x = y = 0$*, ii) If* $\alpha > \theta_0$*, then* $x = y = 1$*.*

**Proof.** The proof is lengthy and is presented in Kultti, Miettunen, Takalo, and Virrankoski (2004). ■

If $u/v$ is small, all the vacancies send wage offers and none of the unemployed workers send wage demands. If $u/v$ is large, all the unemployed workers send wage demands and none of the vacancies send wage offers. The idea of the proof is the following: Assume that $\alpha > \theta_0$, and that all the vacancies send wage offers and none of the unemployed send wage demands. It can be shown that there exists a deviating coalition of vacancies and unemployed such that all deviators would be better off in a market where unemployed send wage demands and none of the vacancies sends offers. Then, the original market cannot be an equilibrium. On the other hand, if $\alpha > \theta_0$ and all the unemployed send wage demands and none of the vacancies sends offers, a deviating coalition does not exist. Unemployed would prefer the new market where vacancies send offers only if the Poisson parameter in the new market is large enough, whereas vacancies prefer the new market only if the Poisson parameter is small enough. It can be shown that if $\alpha > \theta_0$, the required regions for the Poisson parameter do not overlap, thus a deviating coalition cannot exist. If $\alpha < \frac{1}{\theta_0}$, an analogous reasoning applies.

[2] A model by Kultti, Miettunen, Takalo and Virrankoski (2004) considers buyers' and sellers' decisions to wait or search, with auction and bargaining as alternative trading mechanisms. It turns out that the model with auction is utilitywise the same as the wage offer model presented here; also the fractions of staying and moving agents are the same as given by formulae (14) and (15) above.

If there are a lot of unemployed compared to vacancies, Proposition 3 implies that the wage offer density function and the density function for realised wages are decreasing, whereas in case of relatively numerous vacancies, the density functions are increasing.

# 5 Stability of the Equilibrium Market Structure

We can interpret the population shares $x$ and $y$ as strategies of the entering unemployed workers and vacancies, that is, as the probabilities of going to the vacancy market. The probabilities of going to the worker market are $1-x$ and $1-y$. Proposition 2 above and Lemma 1 below show that the model has three equilibrium market structures if $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$. The selection between equilibria is modelled by using an evolutionary argument. Outside equilibrium the agents behave myopically and go to the market where their type fared best in the previous period. The adjustment process is differential, and formalisable by replicator dynamics (see e.g. Lu and McAfee, 1996).[3] To define replicator dynamics let us first establish notation for the unemployed workers' and vacancies' average expected utilities $U$ and $V$, given population shares $x$ and $y$: $U = xU_s + (1-x)U_r$ and $V = yV_r + (1-y)V_s$. In the replicator dynamics the population shares are determined by the following differential equations:

$$\frac{dx}{dt} = x(U_s - U) = x(1-x)(U_s - U_r) \tag{21}$$

$$\frac{dy}{dt} = y(V_r - V) = y(1-y)(V_r - V_s). \tag{22}$$

**Definition 1** *An equilibrium $(x, y)$ is evolutionarily stable if there exists a neighbourhood of $(x, y)$ where the replicator dynamics converges to the equilibrium.*

The replicator dynamics can be easily performed graphically. In Figure 1 we have drawn the equilibrium curves $VE$ and $UE$ on where, by equations (21) and (22), $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. Next we determine the positions of these curves in $(x, y)$-space. First we state

[3] Althoug this is a static (one-shot) model, we can still use replicator dynamics. Instead of assuming agents who live many periods, we assume consecutive generations of one-period-living agents. Or, in a dynamic model, the reservation values are discounted, and the discount factor approaches zero.

**Lemma 1** *Equilibrium curves $VE$ and $UE$ go through* $(0,0)$ *and* $(1,1)$.

**Proof.** In Appendix 1. ■

We have thus shown that $(0,0)$ and $(1,1)$ are pure strategy equilibria. A mixed-strategy equilibrium where $x \in (0,1)$ and $y \in (0,1)$ is economically meaningful only if it is stable. The uniqueness and stability of a mixed-strategy equilibrium is studied in $(x, y)$ -plane (see Figure 1). Along vacancies' equilibrium $VE$, we have $e^{-\phi} = 1 - e^{-\theta} - \theta e^{-\theta}$. The respective equilibrium condition for unemployed, $UE$, is $e^{-\theta} = 1 - e^{-\phi} - \phi e^{-\phi}$. Parameter $\theta = \frac{xu}{yv}$ governs the arrival of workers' applications to vacancies, whereas workers receive vacancies' offers governed by parameter $\phi = \frac{(1-y)\,v}{(1-x)\,u}$ . Denote $\alpha \equiv \frac{u}{v}$.

In order to solve the uniqueness and stability of a mixed-strategy equilibrium we determine the positions of $VE$ and $UE$. When differentiating $VE$ and $UE$ with respect to $x$ and $y$, we use the following results: $\frac{\partial \theta}{\partial x} = \frac{\alpha}{y}$, $\frac{\partial \theta}{\partial x} = \frac{-\alpha x}{y^2}$, $\frac{\partial \phi}{\partial x} = \frac{\phi}{1-x}$, and $\frac{\partial \phi}{\partial y} = -\frac{1}{(1-x)\,\alpha}$. Differentiating $VE$ with respect to $x$ and $y$ yields

$$\frac{dy}{dx}\mid_{VE} = \frac{\phi e^{-\phi}\alpha y + \theta e^{-\theta}\alpha^2 (1-x)}{e^{-\phi}y + \theta^2 e^{-\theta}\alpha (1-x)}, \tag{23}$$

and along $UE$,

$$\frac{dy}{dx}\mid_{UE} = \frac{e^{-\theta}\alpha^2 (1-x) + \phi^2 e^{-\phi}\alpha y}{\theta e^{-\theta}\alpha (1-x) + \phi e^{-\phi} y}. \tag{24}$$

Both equilibrium curves have a positive slope. Waiting firms fare equally well as moving firms only if an increase in the share of moving workers is accompanied with an increase in the share of waiting firms. The same kind of intuiton applies for workers' equilibrium condition, too.

Next we look whether $VE$ is steeper than $UE$ in equilibrium, or the other way round. Subtracting the right-hand side of (24) from that of (23) yields, after a few steps, that $sign\left(\frac{dy}{dx}\mid_{VE} - \frac{dy}{dx}\mid_{UE}\right) = sign\left(2\theta\phi - \theta^2\phi^2 - 1\right)$. In equilibrium $\theta = \phi = \theta_0 \approx 1.146$. Function $2x^2 - x^4 - 1$ has a unique maximum of zero at $x = 1$, therefore $2\theta_0^2 - \theta_0^4 - 1 < 0$, which indicates that in equilibrium $UE$ is steeper than $VE$.

In studying the stability of the mixed-strategy equilibrium, we compare the utility from waiting and moving for firms and workers when they are off the equilibrium curve.

The difference of utilities of waiting and moving for firms is $V_r - V_s = 1 - e^{-\theta} - \theta e^{-\theta} - e^{-\phi}$. Suppose that a firm is on $VE$, and then the fraction of moving workers, $x$, increases. Then

$$\frac{\partial (V_r - V_s)}{\partial x} = \frac{\partial \left(1 - e^{-\theta} - \theta e^{-\theta} - e^{-\phi}\right)}{\partial x} = \frac{\theta e^{-\theta}\alpha}{y} + \frac{\phi e^{-\phi}}{1-x} > 0, \tag{25}$$

which indicates that for a firm, it is now more profitable to wait than move, and therefore the fraction of waiting firms, $y$, will increase. For workers, $U_r - U_s = 1 - e^{-\phi} - \phi e^{-\phi} - e^{-\theta}$. If a worker is on $UE$, and then $y$ increases, the utility difference changes by

$$\frac{\partial (U_r - U_s)}{\partial x} = \frac{\partial \left(1 - e^{-\phi} - \phi e^{-\phi} - e^{-\theta}\right)}{\partial y} = -\frac{\phi e^{-\phi}}{(1-x)\,\alpha} - \frac{e^{-\theta}\alpha x}{y^2} < 0. \tag{26}$$

If the fraction of waiting firms increases, waiting becomes less appealing for workers compared to moving, therefore $x$ will increase. In $(x, y)$ -plane, $y$ decreases above $VE$ and increases below it, and $x$ increase on the left of $UE$ and decreases on the right of it. We thus have

**Proposition 4** *The mixed-strategy equilibrium where $x \in (0,1)$ and $y \in (0,1)$ is evolutionarily stable.*

Lemma 1 tells us that $x = y = 0$ or $x = y = 1$ are also equilibria, by 23 and 24 we know that $VE$ and $UE$ are increasing, and by Proposition 2 we know that at $x \in (0,1)$ and $y \in (0,1)$ they have a unique intersection. If $x = y = 1$, all unemployed workers send applications to vacancies, and none of the vacancies send offers to unemployed workers, and if $x = y = 0$, vice versa. However, we know that those equilibria are necessarily unstable because the equilibrium with $x \in (0,1)$ and $y \in (0,1)$ is stable.

# 6 Aggregate Distribution of Wages

Vacancies and unemployed workers draw their wage offers and demands from distributions $H(w)$ and $F(w)$ which are unobserved. The realised distributions (that are observed) differ from the offers and demands because waiting vacancies hire the worker who has demanded the lowest wage, and waiting workers accept the highest offer. Denote the cumulative distribution of realised wages by $G(w)$ in the vacancy market, and by $M(w)$ in the job seeker market. We have

$$\begin{aligned} M(w) &= \frac{\sum_{k=1}^{\infty} \frac{\phi^k e^{-\phi}}{k!} (H(w))^k}{1-e^{-\phi}} \\ &= \frac{e^{-\phi(1-H(w))} - e^{-\phi}}{1-e^{-\phi}} \\ &= \frac{e^{-\phi}\left(\frac{1}{1-w} - 1\right)}{1-e^{-\phi}}. \end{aligned} \tag{27}$$

That is, $M(w)$ is the probability that the highest offer, conditional on the job seeker receiving at least one offer, is equal to or less than $w$. The denominator $1-e^{-\phi}$ conditions for receiving at least one offer. The density function is

$$m(w) = M'(w) = \frac{e^{-\phi}}{(1-e^{-\phi})(1-w)^2}. \tag{28}$$

In the vacancy market the probability of having $w$ as the lowest wage demand received by a vacancy, conditional on receiving at least one application, is

$$\begin{aligned} G(w) &= \frac{\sum_{k=1}^{\infty} \frac{\theta^k e^{-\theta}}{k!} \left[1-(1-F(w))^k\right]}{1-e^{-\theta}} \\ &= \frac{1-e^{-\theta} - e^{-\theta}\sum_{k=1}^{\infty} \frac{\theta^k (1-F(w))^k}{k!}}{1-e^{-\theta}} \\ &= \frac{1-e^{-\theta} - e^{-\theta}\left(e^{\theta(1-F(w))} - 1\right)}{1-e^{-\theta}} \\ &= \frac{1-e^{-\theta F(w)}}{1-e^{-\theta}}, \end{aligned} \tag{29}$$

where the denominator $1-e^{-\theta}$ conditions for receiving at least one application. Using $F(w) = 1 + \frac{\ln w}{\theta}$ we end up with

$$G(w) = \frac{1-\frac{e^{-\theta}}{w}}{1-e^{-\theta}}. \tag{30}$$

The density function is

$$g(w) \equiv G'(w) = \frac{e^{-\theta}}{(1-e^{-\theta})\, w^2}. \tag{31}$$

The value of $m(w)$ is the probability that $w$ is the highest offer a waiting job seeker gets; that is, $m(w)$ is the probability that his realised wage is $w$. Similarly, the value of $g(w)$ is the probability that $w$ is the lowest wage demand that a waiting vacancy faces. The realised aggregate density function of wages, $r(w)$ is a weighted combination of densities $g(w)$ and $m(w)$ as follows:

$$r(w) = \begin{cases} \dfrac{m(w)(1-x)u}{(1-x)u+yv} & if \quad w \in \left[0, e^{-\theta}\right], \\ \dfrac{m(w)(1-x)u + g(w)yv}{(1-x)u+yv} & if \quad w \in (e^{-\theta}, 1-e^{-\phi}], \\ \dfrac{g(w)yv}{(1-x)u+yv} & if \quad w \in (1-e^{-\phi}, 1]. \end{cases} \tag{32}$$

For $w < e^{-\theta}$, only vacancies send offers, and there are $(1-x)u$ job seekers who receive them. For $w > 1-e^{-\phi}$, only job seekers send offers to $yv$ vacancies. For middle-range wages, $(1-y)v$ vacancies send offers to $(1-x)u$ unemployed workers and $xu$ unemployed workers send wage demands to $yv$ vacancies. We have yet to determine the equilibrium values of $x$ and $y$, which are determined by indifference conditions for vacancies and unemployed workers.

Now we have all the ingredients to determine the realised aggregate wage distribution that emerges in equilibrium where search is two-sided. Using the equilibrium values for $x$ and $y$, from 19 and 20 , and the solutions for the density functions from 28 and 31, gives

**Proposition 5** *The density function of realised wages $r(w)$ satisfies*

$$r(w) = \begin{cases} \dfrac{e^{-\theta_0}(\theta_0-\alpha)}{(1-e^{-\theta_0})(\theta_0-1)(1+\alpha)(1-w)^2} & \text{if } w \in \left[0, e^{-\theta_0}\right], \\ \dfrac{e^{-\theta_0}}{(1-e^{-\theta_0})(\theta_0-1)(1+\alpha)}\left[\dfrac{\theta_0-\alpha}{(1-w)^2} + \dfrac{\alpha\theta_0-1}{w^2}\right] & \text{if } w \in (e^{-\theta_0}, 1-e^{-\theta_0}], \\ \dfrac{e^{-\theta_0}(\alpha\theta_0-1)}{(1-e^{-\theta_0})(\theta_0-1)(1+\alpha)w^2} & \text{if } w \in (1-e^{-\theta_0}, 1]. \end{cases}$$

*where $\alpha \in \left(\dfrac{1}{\theta_0}, \theta_0\right)$ and $\theta_0$ is the solution to equation 16.*

Using $\theta_0 \approx 1.146$ we have

$$r(w) \approx \begin{cases} \dfrac{3.192\,(1.146-\alpha)}{(1+\alpha)\,(1-w)^2} \text{if } w \in [0, 0.317\,91] \\ \dfrac{3.192}{1+\alpha}\left[\dfrac{1.146-\alpha}{(1-w)^2} + \dfrac{1.146\alpha-1}{w^2}\right] \text{if } w \in (0.317\,91, 0.682\,091] \\ \dfrac{3.192\,(1.146\alpha-1)}{(1+\alpha)\,w^2} \text{if } w \in (0.682\,0911, 1]. \end{cases} \tag{33}$$

For $u = v$, Figures 2a and 2b show the density functions of realised wages in the job seeker market and in the vacancy market, respectively. Figure 2c shows the aggretate wage density function. The wage distribution is first increasing, in the end decreasing, and u-shaped in the middle. Thus, we do not get exactly the wage distribution observed empirically but one that still has several desirable features.Figures 3a and 3b show that if the ratio $u/v$ increases, the wage density function shifts to the left. Our model thus predicts that increasing the relative supply of labour decreases the average wage.

# 7 Dynamic Models

The static version of our model is sufficient for the generation of a wage distribution as it looks pretty much the same in the dynamic models. Still, one may be interested in the dynamics, especially if there are data on discount factors or separation rates. This in mind we provide the central results of the dynamic models in this section. The detailed derivation of the results is relegated to the appendix. The analysis mirrors to the most part the analysis of the static case, but the derivation of the equilibrium mixed strategies for wage demands and offers is more complicated. This is because the upper and lower limits of the support of the strategies are now endogenous being determined by the expected life time utilities, while in the static model the outside option, or the expected life time utility, of an agent who rejects an offer is zero.

## 7.1 Dynamic Model 1

Instead of assuming one-period-living agents, we now assume that the agents live infinitely long and discount future at rate $\delta$. The agents send and receive offers each period until they are matched. A matched pair exits the economy and produces an output of unity until infinity. They are replaced by identical but yet unmatched agents (clones). Each vacancy and worker who send wage offers use a symmetric mixed strategy. The fractions of agents in the vacancy market are

$$\begin{aligned} x &= \frac{\theta_0 \left(\alpha\theta_0 - 1\right)}{\alpha\left(\theta_0^2 - 1\right)}, \\ y &= \frac{\alpha\theta_0 - 1}{\theta_0^2 - 1}, \end{aligned}$$

they are the same as in the static model.

**Proposition 6** *In the dynamic model where the exited agents are replaced by clones the aggregate wage density is*

$$r(w) = \begin{cases} \dfrac{m(w)(1-x)\alpha}{(1-x)\alpha + y} & \text{if } a \le w \le A, \\ \dfrac{m(w)(1-x)\alpha + g(w)y}{(1-x)\alpha + y} & \text{if } A < w \le b, \\ \dfrac{g(w)y\alpha}{(1-x)\alpha + y} & \text{if } b < w \le B,. \end{cases}$$

*where* $a = \delta\dfrac{1-e^{-\theta_0} - \delta\theta_0 e^{-\theta_0}}{1-\delta\theta_0 e^{-\theta_0}}$, $b = a/\delta$, $A = \dfrac{e^{-\theta_0}}{1-\delta\theta_0 e^{-\theta_0}}$, $B = \dfrac{1-\delta+\delta e^{-\theta_0}}{1-\delta\theta_0 e^{-\theta_0}}$,

$$m(w) = \frac{(1-\delta)e^{-\theta_0}(1-\delta\theta_0 e^{-\theta_0})}{(1-e^{-\theta_0})\left[(1-w)(1-\delta\theta_0 e^{-\theta_0}) - \delta e^{-\theta}\right]^2},$$

$$g(w) = \frac{\left(1-\delta\theta_0 e^{-\theta_0}\right)(1-\delta)e^{-\theta_0}}{(1-e^{-\theta_0})\left[w(1-\delta\theta_0 e^{-\theta_0}) - \delta e^{-\theta_0}\right]^2},$$

$\alpha \in \left(\dfrac{1}{\theta_0}, \theta_0\right)$ *,and where* $\theta_0$ *satisfies 16,* $\theta_0 \approx 1.146$.

The wage distribution is qualitatively very similar to that in in the static model: low and increasing at low wages, high and decreasing at high wages, and high and u-shaped a tteh middle-range wages.

## 7.2 Dynamic Model 2 (A General Equilibrium Model)

The general equilibrium model described in Section 2 results in the same fractions of agents in the vacancy market as the other two models. For the aggregate wage distribution we have

**Proposition 7** *In the general equilibrium model the aggregate density of wages is*

$$r(w) = \begin{cases} \dfrac{m(w)(1-x)\alpha}{(1-x)\alpha + y} & \textit{if } l_1 \leq w \leq l_2, \\ \dfrac{m(w)(1-x)\alpha + g(w)y}{(1-x)\alpha + y} & \textit{if } \; l_2 < w \leq h_1, \\ \dfrac{g(w)y\alpha}{(1-x)\alpha + y} & \textit{if } \; h_1 < w \leq h_2, \end{cases}$$

*where* $l_1 = \dfrac{\delta(1-b)\left(1 - e^{-\theta_{00}} - \theta_{00}e^{-\theta_{00}}\right)}{1 - \delta(1-b)\theta_{00}e^{-\theta_{00}}}$, $l_2 = \dfrac{e^{-\ \theta_0}}{1 - \delta(1-b)\ \theta_0 e^{-\ \theta_0}}$,

$$h_1 = \frac{1 - \delta(1-b)\theta_{00}e^{-\theta_{00}} - e^{-\theta_{00}}}{1 - \delta(1-b)\theta_{00}e^{-\theta_{00}}}, \; h_2 = \frac{1 - \delta(1-b)\left(1 - e^{-\ \theta_0}\right)}{-\delta(1-b)\ \theta_0 e^{-\ \theta_0}},$$

$$m(w) = \frac{\left(1 - \delta(1-b)\theta_{00}e^{-\theta_{00}}\right)(1 - \delta(1-b))e^{-\theta_{00}}}{(1 - e^{-\theta_{00}})\left[(1-w)(1 - \delta(1-b)\theta_{00}e^{-\theta_{00}}) - \delta(1-b)e^{-\theta_{00}}\right]^2},$$

$$g(w) = \frac{\left(1 - \delta(1-b)\ \theta_0 e^{-\ \theta_0}\right)(1 - \delta(1-b))e^{-\ \theta_0}}{(1 - e^{-\ \theta_0})\left[w(1 - \delta(1-b)\ \theta_0 e^{-\ \theta_0}) - \delta(1-b)e^{-\ \theta_0}\right]},$$

$\alpha \in \left(\dfrac{1}{\theta_0}, \theta_0\right)$ *,and where* $\theta_0$ *satisfies 16,* $\theta_0 \approx 1.146$.

The separation rate $b$ affects the number of unmatched agents, and it also affects $\alpha$, the ratio of unemployed workers to firms. Therefore, for having $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$, $b$ must be in certain limits. For the same reason we also must have a restriction concerning the relative magnitude of $W$ and $E$, the total numbers of workers and firms. A preliminary analysis (not yet shown in this article) shows that a steady state equilibrium with two-sided search and the wage density function given in Proposition 7 exists only if $\frac{W}{E} \in \left(\frac{1}{\theta_0^2}, \theta_0^2\right)$. Note that this range is wider than range $\left(\frac{1}{\theta_0}, \theta_0\right)$ for $u/v$. That is, in a general equilibrium model allows a larger asymmetry in the relative number of workers and firms, while still producing a non-monotonous wage distribution.

# 8 Conclusion

We derive a wage density function for homogenous firms and homogenous workers. Both vacancies and unempoyed workers can send wage demands or offers, which is what we often see happening in real labour markets. We show that the symmetric equilibrium for offers and demands is in mixed strategies, and we solve the equilibrium fractions of vacancies and unemployed workers who are engaged in sending or receiving offers. For small wages the density function of realised wages is low and increasing, for high wages it is low and decreasing, and for middle-range wages it is high and u-shaped. The wage distribution the model produces is not exactly the one observed empirically, but it is fairly close to that. It is notable that we get this distribution without assuming any kind of heterogeneity among workers or firms. There are several equilibria in the model but of them only the evolutionarily stable one produces the interesting wage distribution, whereas the equilibria that are associated with monotonous distributions are unstable. Our model also predicts that an increase in the relative supply of labour decreases the average wage. Another interesting result is that the mixed strategies that vacancies and job seekers use are utilitywise equivalent to auctions where the agents know the number of their competitors.

We believe that our approach offers plenty of chances for applications and generalisations. The meeting technology we use means that the matching function is well determined with a firm microfoundation. Consequently, one can do rigorous comparative statics as nothing comes outside of the model. In particular, one can determine the response of duration of unemployment spells when the measure of workers or firms changes, or when the expected life-span of matches change. The model is well suited to consider the implications of worker/firm heterogeneity on wage distribution. In the next version of the paper we will continue the analysis of the dynamic model 2.

We think that the results of this article nicely illuminate the strengths of the urn-ball model over the search models. In the end, it is clear that whatever one can do using the search models, one can also do using the urn-ball models, and with the latter ones one can do much more, with no need to postulate the black box of a matching function. To give an example outside this article, it is relatively straightforward to consider a situation where vacancies post wages that are observed by the unemployed who strategically decide which vacancy to contact based on the observed wage offers (see e.g. Kultti 1999; Julien, Kennes and King, 2001). This is practically impossible in the search models. Against this background it is somewhat a mystery to us why search models are still used.

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# Appendix

## A1 Proof of Lemma 1

Vacancies' equilibrium $VE$ can be written as

$$1 - e^{-\frac{xu}{yv}} - \frac{xu}{yv} e^{-\frac{xu}{yv}} - e^{-\frac{(1-y)\,v}{(1-x)\,u}} = 0,$$

and unemployed workers' equilibrium $UE$ is

$$1 - e^{-\frac{(1-y)\,v}{(1-x)\,u}} - \frac{(1-y)\,v}{(1-x)\,u} e^{-\frac{(1-y)\,v}{(1-x)\,u}} - e^{-\frac{xu}{yv}} = 0.$$

The behaviour of $VE$ and $UE$ near $(0,0)$ is analyzed first.

1. $(x,y) \to (0,y)$ . i) $VE$ becomes $-e^{-\frac{(1-y)\,v}{u}} = 0$, which cannot hold for any $y \in [0,1]$. ii) $UE$ becomes $-e^{-\frac{(1-y)\,v}{u}} - \frac{(1-y)\,v}{u} e^{-\frac{(1-y)\,v}{u}} = 0$, which cannot hold for any $y \in [0,1]$.

2. $(x,y) \to (x,0)$. i) $VE$ becomes $1 - e^{-\frac{v}{(1-x)\,u}} = 0$, which cannot hold for any $x \in [0,1]$. ii) $UE$ becomes $1 - e^{-\frac{v}{(1-x)\,u}} - \frac{v}{(1-x)\,u} e^{-\frac{v}{(1-x)\,u}} = 0$, which does not hold for any $x \in [0,1]$.

Clearly, neither $VE$ nor $UE$ cannot go through $(0,y)$ or $(x,0)$, so they must go through $(0,0)$. We check that this is possible. Assume that along $VE$, $x/y \to a$ as $x \to 0$ and $y \to 0$. Then

$$\lim_{x\to 0, y\to 0} \frac{1-y}{1-x} = \lim \left( \frac{1}{1-x} - \frac{1}{\frac{1}{y} - \frac{x}{y}} \right) = 1 - \lim \frac{1}{\frac{1}{y} - a} = 1.$$

Then

$$\lim_{x\to 0, y\to 0} \left(1 - e^{-\theta} - \theta e^{-\theta} - e^{-\phi}\right) = 1 - e^{-a\alpha} - a\alpha e^{-a\alpha} - e^{-1/\alpha},$$

which, by for example letting $\alpha = 1$, equals zero if $a = 1.285$. For any other $\alpha > 0$ one can find $a > 0$ that satisfies $VE$ going through $(0,0)$. Assume that along $UE$, $x/y \to b$ as $x \to 0$ and $y \to 0$. Then

$$\lim_{x\to 0, y\to 0} \left(1 - e^{-\phi} - \phi e^{-\phi} - e^{-\theta}\right) = 1 - e^{-1/\alpha} - \frac{1}{\alpha} e^{-1/\alpha} - e^{-b\alpha},$$

which equals zero if $\alpha = 1$ and $b = 1.33$. Because $a < b$, $VE$ is above $UE$ near $(0,0)$, which is does not contradict $UE$ being steeper than $VE$ in their intersection at strictly positive $x$ and $y$.

Next check the curves' positions near $(1,1)$.

1. $(x,y) \to (x,1)$ . i) $VE$ becomes $-e^{-\frac{xu}{v}} - \frac{xu}{v}e^{-\frac{xu}{v}} = 0$, which cannot hold for any $x \in [0,1]$. ii) $UE$ becomes $-e^{-\frac{xu}{v}} = 0$, which cannot hold for any $x \in [0,1]$.

2. $(x,y) \to (1,y)$ . i) $VE$ becomes $1 - e^{-\frac{u}{yv}} - \frac{u}{yv}e^{-\frac{u}{yv}} = 0$, which cannot hold for any $y \in [0,1]$. ii) $UE$ becomes $1 - e^{-\frac{u}{yv}} = 0$, which cannot hold for any $y \in [0,1]$.

We see that $VE$ and $UE$ must go through $(1,1)$. As $x$ and $y$ approach 1, assume that $\frac{1-y}{1-x} \to c$ along $VE$ and $\frac{1-y}{1-x} \to g$ along $UE$. Then $VE$ becomes

$$\lim_{x\to 1, y\to 1}\left(1 - e^{-\theta} - \theta e^{-\theta} - e^{-\phi}\right) = 1 - e^{-\alpha} - \alpha e^{-\alpha} - e^{-c/\alpha} = 0,$$

which holds for example if $\alpha = 1$ and $c = 1.33$. In the limit $UE$ equals

$$\lim_{x\to 1, y\to 1}\left(1 - e^{-\phi} - \phi e^{-\phi} - e^{-\theta_0}\right) = 1 - e^{-g/\alpha} - \frac{g}{\alpha}e^{-g/\alpha} - e^{-\alpha} = 0;$$

with $\alpha = 1$ it holds if $g = 1.285$. Near $(1,1)$, $UE$ lies above $VE$, which is consistent with $UE$ being steeper than $VE$ in their intersection at strictly positive $x$ and $y$.

## A2 Dynamic Model 1

### A2.1 Firms send offers to workers

Let us study a situation where the workers are like urns and employers as balls. Our aim is to determine the mixed strategy of the employers in a dynamic model focusing on a steady-state. It turns out that the agents' expected utilities are the same as in a corresponding model where the wages are determined by auction. We need the results of the auction to determine the mixed strategies, and that in mind we first determine the agents' expected utilities under auction. The workers' and the employers' utilities are determined by the following equations

$$V_w = (e^{-\phi} + \phi e^{-\phi})\delta V_w + (1 - e^{-\phi} - \phi e^{-\phi})(1 - \delta V_e), \tag{34}$$

$$V_e = e^{-\phi}(1-\delta V_w) + (1-e^{-\phi})\delta V_e. \tag{35}$$

In 34, $e^{-\phi}$ is the probability that no firm comes to the worker, and $\phi e^{-\phi}$ stands for the probability of just one firm arriving, in which case the firm makes a take-it-or-leave-it offer. In both these cases, the worker continues to the next period with his discounted reservation value $\delta V_w$. If he gets two or more firms, the firms engage in Bertrand competition for the right to employ the worker. The firms, regardless of which of them employs the worker, get their discounted reservation value $\delta V_e$, and the worker gets $1-\delta V_e$. In 35, with probability $e^{-\phi}$ the firm is the only one that meets the worker, the firm makes take-it-or-leave-it offer and gets one minus the worker's discounted reservation value. If the firms has at least one competitor, it gets it discounted reservation value. From these one gets explicit expressions

$$V_w = \frac{1-e^{-\phi}-\phi e^{-\phi}}{1-\delta\phi e^{-\phi}}, \tag{36}$$

$$V_e = \frac{e^{-\phi}}{1-\delta\phi e^{-\phi}}. \tag{37}$$

These utilities are the same as in Kultti (1999), except for the slightly different way of discounting.

Let us now leave the auction and assume that employers use a continuous mixed strategy $H$ with support $[a,b]$. An employer's expected utility when he offers wage $w \in [a,b]$ is given by

$$V_e(w) = \sum_{k=0}^{\infty} e^{-\phi}\frac{\phi^k}{k!}\left[H^k(w)(1-w) + \left(1-H^k(w)\right)\delta V_e\right], \tag{38}$$

which after some simplification equals

$$V_e(w) = (1-w)e^{-\phi\left(1-H^k(w)\right)} + \delta V_e\left(1-e^{-\phi\left(1-H^k(w)\right)}\right). \tag{39}$$

Next we use the fact that any wage in the support of the mixed strategy yields the same utility to the employer, in particular, this holds for the lowest and the highest wages

$$V_e = V_e(a) = (1-a)e^{-\phi} + \delta V_e(1-e^{-\phi}) = V_e(b) = 1-b. \tag{40}$$

From this we can solve for

$$b = 1-(1-a)e^{-\phi} - \delta V_e(1-e^{-\phi}) = 1-e^{-\phi} + \delta V_w e^{-\phi} - \delta V_e(1-e^{-\phi}), \tag{41}$$

where the last equality is based on the fact that the lowest wage in the support of the mixed strategy must equal the workers' outside option, i.e. it must make them indifferent between accepting it and continuing search. Thus, we have that

$$a = \delta V_w. \tag{42}$$

We let $h(w) = H'(w)$ and determine the workers' expected utility

$$V_w = \sum_{k=1}^{\infty} e^{-\phi} \frac{\phi^k}{k!} \int_a^b wkh(w)H^{k-1}(w)dw + e^{-\phi}\delta V_w \tag{43}$$

which equals

$$V_w = \sum_{k=1}^{\infty} e^{-\phi} \frac{\phi^k}{k!} \left[ \Big/_{a}^{b} H^k(w)w - \int_a^b H^k(w)dw \right] = \sum_{k=1}^{\infty} e^{-\phi} \frac{\phi^k}{k!} \left[ b - \int_a^b H^k(w)dw \right] \tag{44}$$

which in turn equals, using Fubini's theorem,

$$V_w = b - \int_a^b e^{-\phi(1-H(w))}dw. \tag{45}$$

Inserting this and (40) into (41) yields

$$b = 1 - e^{-\phi} - \frac{\delta}{1-\delta} e^{-\phi} \int_a^b e^{-\phi(1-H(w))}dw. \tag{46}$$

Next we impose that the expected utility of an employer equals that of an employer with auction

$$V_e = V_e(b) = 1 - b = \frac{e^{-\phi}}{1 - \delta\phi e^{-\phi}} \tag{47}$$

which yields the following formula:

$$b = \frac{1 - e^{-\phi} - \delta\phi e^{-\phi}}{1 - \delta\phi e^{-\phi}}. \tag{48}$$

From (40) we get by similarly

$$V_e = V_e(a) = \left(1 - \delta + \delta e^{-\phi}\right)^{-1} (1-a)e^{-\phi} = \frac{e^{-\phi}}{1 - \delta\phi e^{-\phi}}. \tag{49}$$

From this we can solve

$$a = \delta V_w = \delta \frac{1 - e^{-\phi} - \delta\phi e^{-\phi}}{1 - \delta\phi e^{-\phi}} = \delta b. \tag{50}$$

Using the fact that $V_e(w) = V_e(b)$, solving $V_e(w)$ from (39) and equating it with (47) yields

$$e^{-\phi(1-H(w))}\left[1 - w - \delta\frac{e^{-\phi}}{1-\delta\phi e^{-\phi}}\right] = (1-\delta)\frac{e^{-\phi}}{1-\delta\phi e^{-\phi}}. \tag{51}$$

From this we can solve the equilibrium mixed strategy

$$H(w) = 1 - \frac{1}{\phi}\ln\left(1 - w - \delta\frac{e^{-\phi}}{1-\delta\phi e^{-\phi}}\right) + \frac{1}{\phi}\ln\left((1-\delta)\frac{e^{-\phi}}{1-\delta\phi e^{-\phi}}\right). \tag{52}$$

The equilibrium mixed strategy is unobservable while the realised wages that result from it generate an observable wage distribution. We denote the cumulative distribution function for realised wages by $M$ and the corresponding density function is denoted by $m$. Let us determine the probability that wage $w$ is observed.

$$\left(1 - e^{-\phi}\right) m(w) = \sum_{k=1}^{\infty} e^{-\phi}\frac{\phi^k}{k!}kh(w)H^{k-1}(w) = \phi e^{-\phi(1-H(w))}h(w) \tag{53}$$

From this we get

$$m(w) = \frac{e^{-\phi(1-H(w))} - e^{-\phi}}{1-e^{-\phi}} \tag{54}$$

Inserting (52) above and manipulating a little yields an explicit formula

$$m(w) = \frac{(1-\delta)e^{-\phi}(1-\delta\phi e^{-\phi})}{(1-e^{-\phi})\left[(1-w)(1-\delta\phi e^{-\phi}) - \delta e^{-\phi}\right]^2}. \tag{55}$$

### A2.2 Workers send applications to firms

In the standard auction model the utilities of firms and workers are

$$V_e = (e^{-\theta} + \theta e^{-\theta})\delta V_e + (1 - e^{-\theta} - \theta e^{-\theta})(1 - \delta V_w), \tag{56}$$

$$V_w = e^{-\theta}(1 - \delta V_w) + (1 - e^{-\theta})\delta V_w. \tag{57}$$

Solving these yields

$$V_e = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - \delta\theta e^{-\theta}}, \tag{58}$$

$$V_w = \frac{e^{-\theta}}{1 - \delta\theta e^{-\theta}}. \tag{59}$$

Next assume that workers use a continuous mixed strategy $F$ with support $[A, B]$ .A worker's expected utility when he asks wages $w \in [A, B]$ is

$$\begin{aligned} V_w(w) &= \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} \left( [1 - F(w)]^k w + \left(1 - [1 - F(w)]^k\right) \delta V_w \right) \\ &\Leftrightarrow V_w(w) = w e^{-\theta F(w)} + \left(1 - e^{-\theta F(w)}\right) \delta V_w \\ &\Leftrightarrow V_w = \frac{w e^{-\theta F(w)}}{1 - \delta + \delta e^{-\theta F(w)}}. \end{aligned} \tag{60}$$

Any wage in support $[A, B]$ yields the same utility to a worker, especially $V_w(A) = V_w(B)$:

$$V_w(A) = A e^{-\theta F(A)} + \left(1 - e^{-\theta F(A)}\right) \delta V_w = B e^{-\theta F(B)} + \left(1 - e^{-\theta F(A)}\right) \delta V_w = V_w(B), \tag{61}$$

and using $F(A) = 0$ and $F(B) = 1$ we have

$$V_w = A, \tag{62}$$

$$A = B e^{-\theta} + \left(1 - e^{-\theta}\right). \tag{63}$$

The highest offer the worker makes must leave the firm its reservation value:

$$B = 1 - \delta V_e. \tag{64}$$

and $A$ can be written as

$$A = (1 - \delta V_e) + \left(1 - e^{-\theta}\right) \delta V_w. \tag{65}$$

Let $f(w) \equiv F'(w)$ and determine a firm's expected utility as

$$V_e = \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{k!} \int_A^B (1-w) f(w) k [1-F(w)]^{k-1} dw + e^{-\theta}\delta V_e \quad (66)$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{k!} \left( -[1-F(B)]^k + [1-F(A)]^k \right)$$

$$- \int_A^B \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{k!} f(w) k [1-F(w)]^{k-1} w dw + e^{-\theta}\delta V_e$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta}{k!} - \int_A^B \sum_{k=1}^{\infty} \frac{e^{-\theta}\theta^k}{(k-1)!} f(w) [1-F(w)]^{k-1} w dw + e^{-\theta}\delta V_e$$

$$= 1 - e^{-\theta} - \theta \int_A^B f(w) w \sum_{k=0}^{\infty} \frac{e^{-\theta}\theta^k [1-F(w)]^k}{k!} dw + e^{-\theta}\delta V_e$$

$$= 1 - e^{-\theta} - \theta \int_A^B e^{-\theta F(w)} f(w) w dw + e^{-\theta}\delta V_e$$

$$= 1 - e^{-\theta} + Be^{-\theta F(B)} - Ae^{-\theta F(A)} - \int_A^B e^{-\theta F(w)} dw + e^{-\theta}\delta V_e$$

$$= 1 - e^{-\theta} + Be^{-\theta} - A - \int_A^B e^{-\theta F(w)} dw + e^{-\theta}\delta V_e.$$

Then subsitute $1-\delta V_e$ for $B$ and rearrange to have

$$V_e = 1 - A - \int_A^B e^{-\ \theta F(w)} dw. \quad (67)$$

Using (66) and $A = V_w$ in (64) we get

$$A = \left[ 1 - \delta \left( 1 - A - \int_A^B e^{-\theta_0 F(w)} dw \right) \right] e^{-\theta_0} + \delta \left( 1 - e^{-\theta_0} \right) A \quad (68)$$

which implies that

$$A = \frac{1 - \delta e^{-\theta_0} + \delta e^{-\theta_0} \int_A^B e^{-\theta_0 F(w)} dw}{1-\delta}. \quad (69)$$

The lower bound of the support of wage asks cannot be determined explicitly.

Next we impose that the expected utility of a worker equals that of a worker with auction:

$$V_w = \frac{e^{-\theta_0}}{1 - \delta\theta_0 e^{-\theta_0}} = A. \quad (70)$$

Utilizing (60) yields

$$V_w = Be^{-\theta_0} + \delta V_w \left( 1 - e^{-\theta_0} \right), \quad (71)$$

which gives, along with (70), that

$$V_w = \frac{Be^{-\theta_0}}{1-\delta+\delta e^{-\theta_0}} = \frac{e^{-\theta_0}}{1-\delta\theta_0 e^{-\theta_0}}. \tag{72}$$

Solving for $B$ gives

$$B = \frac{1-\delta+\delta e^{-\theta}}{1-\delta\theta e^{-\theta}}. \tag{73}$$

Equating $V_w$ given by (60) and that given by (72) yields

$$\frac{we^{-\theta F(w)}}{1-\delta+\delta e^{-\theta F(w)}} = \frac{e^{-\theta}}{1-\delta\theta e^{-\theta}} \tag{74}$$

which implies that

$$e^{-\theta F(w)} = \frac{\dfrac{(1-\delta)\, e^{-\theta}}{1-\delta\theta e^{-\theta}}}{w - \dfrac{\delta e^{-\theta}}{1-\delta\theta e^{-\theta}}} \tag{75}$$

Taking logarithms and arranging results in

$$F(w) = \frac{1}{\theta}\left[\ln\left(w - \frac{\delta e^{-\theta}}{1-\delta\theta e^{-\theta}}\right) - \ln\left(\frac{(1-\delta)\, e^{-\theta}}{1-\delta\theta e^{-\theta}}\right)\right]. \tag{76}$$

We denote again the cumulative distribution function for realised wages by $G(w)$ and the corresponding density function is denoted by $g(w)$.

$$G(w) = \frac{1-e^{-\ \theta F(w)}}{1-e^{-\ \theta}} \tag{77}$$

$$= \frac{1}{1-e^{-\ \theta}} - \frac{(1-\delta)e^{-\ \theta}}{(1-e^{-\ \theta})\left[w(1-\delta\ \theta e^{-\ \theta}) - \delta e^{-\ \theta}\right]}. \tag{78}$$

The density function is

$$g(w) = \frac{\left(1-\delta\ \theta e^{-\ \theta}\right)(1-\delta)e^{-\ \theta}}{(1-e^{-\ \theta})\left[w(1-\delta\ \theta e^{-\ \theta}) - \delta e^{-\ \theta}\right]^2}. \tag{79}$$

### A2.3 Equilibrium Market Structure

In equilibrium, workers are indifferent between sending applications and receiving offers from firms, and firms are indifferent between making offers and receiving applications from workers. The equilibrium condition for a firm, $VE$, is

$$\frac{e^{-\phi}}{1-\delta\phi e^{-\phi}} = \frac{1-e^{-\ \theta}-\ \theta e^{-\ \theta}}{1-\delta\ \theta e^{-\ \theta}}, \tag{80}$$

and the respective condition $UE$ for a worker is

$$\frac{e^{-\ \theta}}{1-\delta\ \theta e^{-\ \theta}}=\frac{1-e^{-\phi}-\phi e^{-\phi}}{1-\delta\phi e^{-\phi}}. \tag{81}$$

Proceeding as in the static case, it turns out that in equilibrium $\theta=\phi=\theta_0\approx 1.146$, and

$$x = \frac{\theta_0\left(\alpha\theta_0-1\right)}{\alpha\left(\theta_0^2-1\right)}, \tag{82}$$

$$y = \frac{\alpha\theta_0-1}{\theta_0^2-1}. \tag{83}$$

as in the static model. Along $UE$,

$$\frac{dy}{dx}\mid_{UE}=\frac{C\dfrac{d\phi}{dx}-A\dfrac{d\theta}{dx}}{A\dfrac{d\theta}{dy}-C\dfrac{d\phi}{dy}}, \tag{84}$$

where $A=\dfrac{-e^{-\theta_0}}{\left(1-\delta\theta_0 e^{-\theta_0}\right)^2}$ and $C=\dfrac{e^{-\phi}\left[(1-\delta)\phi+\delta\left(1-e^{-\phi}\right)\right]}{\left(1-\delta\phi e^{-\phi}\right)^2}$. Along $VE$,

$$\frac{dy}{dx}\mid_{VE}=\frac{B\dfrac{d\theta_0}{dx}-D\dfrac{d\phi}{dx}}{D\dfrac{d\phi}{dy}-B\dfrac{d\theta_0}{dy}}, \tag{85}$$

where $B=\dfrac{e^{-\theta_0}\left[(1-\delta)\theta+\delta\left(1-e^{-\theta}\right)\right]}{\left(1-\delta\theta e^{-\theta}\right)^2}$ and $D=\dfrac{-e^{-\phi}}{\left(1-\delta\phi e^{-\phi}\right)^2}$ . Curve $UE$ is steeper than $VE$ if $(AD-BC)\left(\dfrac{d\phi}{dx}\dfrac{d\theta}{dy}-\dfrac{d\theta}{dx}\dfrac{d\phi}{dy}\right)>0$. In equilibrium $\phi=\theta=\theta_0\approx 1.146$, and the sign of $AD-BC$ turns out to be equal to the sign of $(1-\delta)\left(1-\theta_0\right)$, which is negative. Expression $\dfrac{d\phi}{dx}\dfrac{d\theta}{dy}-\dfrac{d\theta}{dx}\dfrac{d\phi}{dy}$ simplifies to $\dfrac{y-x}{(1-x)^2\,y^2}$, which is negative by $x>y$ in equilibrium. These results yield

**Proposition A2.1** *In a dynamic model where the agents who exit are replaced by unmatched clones, the mixed-strategy equilibrium where $x\in(0,1)$ and $y\in(0,1)$ is stable.*

As in the static case, the endpoints of both $VE$ and $UE$ are at $(0,0)$ and at $(1,1)$.

### A2.4 Distribution of Realised Wages

Calculating the distribution of realised wages goes analogously to the corresponding task in the static model:

$$r(w)=\begin{cases}\dfrac{m(w)(1-x)\alpha}{(1-x)\alpha+y} & \text{if } a\leq w\leq A\\ \dfrac{m(w)(1-x)\alpha+g(w)y}{(1-x)\alpha+y} & \text{if } A<w\leq b\\ \dfrac{g(w)y\alpha}{(1-x)\alpha+y} & \text{if } b<w\leq B\end{cases}\tag{86}$$

where $a=\delta\dfrac{1-e^{-\phi}-\delta\phi e^{-\phi}}{1-\delta\phi e^{-\phi}}$, $b=a/\delta$, $A=\dfrac{e^{-\theta}}{1-\delta\theta e^{-\theta}}$ and $B=\dfrac{1-\delta+\delta e^{-\theta}}{1-\delta\theta e^{-\theta}}$.

## A3 Dynamic Model 2 (General Equilibrium Model)

### A3.1 Firms send offers to workers

Assume that the total number of workers is $W$ and the total number of employers is $E$. Some of them are matched with each other in productive activities, while others are looking for a partner. The number of unemployed is denoted by $u$ and the number of vacancies by $v$. Procuction happens in pairs, therefore

$$W-u=E-v.\tag{87}$$

For the moment we focus just on the matching market which is assumed to be in a steady state. The only complication to the standard set-up is an exogenous separation probability $b$, and the fact that a worker who is employed at wage $w$ gets the wage each period as long as the employment relationship lasts, and correspondingly the employer get $1-w$ each period.

#### A3.1.1 Auction

Let the unemployed be urns and the vacancies balls, and let $\phi=\frac{v}{u}$. First we determine their expected life time utilities when wages are determined in auction. The timing is as follows: We determine the expected life time utility of an unemployer worker and a vacancy at the very beginning of a period. The utility of a worker or an employer that has a partner is evaluated right after that, i.e. within the same period before anything else happens. After that the parties produce and get their shares of the production. After that separations take place. The utility of an unemployed worker is determined by

$$A_u=\left(e^{-\phi}+\phi e^{-\phi}\right)\delta A_u+\left(1-e^{-\phi}-\phi e^{-\phi}\right)A_u(\overline{w})\tag{88}$$

where $\overline{w}$ is the wage that vacancies offer when there are two or more vacancies competing for a worker. We take it as given for now, and determine the equilibrium value later on. We have also used the fact that when a worker meets exactly one vacancy the vacancy makes a take-it-or-leave-it offer that leaves no surplus to the worker. The utility of a matched worker with wage $\overline{w}$ is determined by

$$A_u(\overline{w}) = \overline{w} + \delta b A_u + \delta(1-b)A_u(\overline{w}). \tag{89}$$

The expected utility of a vacancy is determined by

$$A_v = e^{-\phi} A_v (1 - \underline{w}) + \left(1 - e^{-\phi}\right) \delta A_v \tag{90}$$

where $\underline{w}$ is the wage that a vacancy offers when it gets to make a take-it-or-leave-it offer. The expected utility of an employer who employs at wage $\underline{w}$ is determined by

$$A_v(1-\underline{w}) = 1 - \underline{w} + \delta b A_v + \delta(1-b)A_v(1-\underline{w}). \tag{91}$$

From these equations we can determine the expected utilities as the function of the wages

$$A_u = \frac{\left(1 - e^{-\phi} - \phi e^{-\phi}\right) \overline{w}}{(1-\delta)\left(1 - \delta(1-b)\left(e^{-\phi} + \phi e^{-\phi}\right)\right)} \tag{92}$$

$$A_u(\overline{w}) = \frac{\left(1 - \delta(e^{-\phi} + \phi e^{-\phi})\right) \overline{w}}{(1-\delta)\left(1 - \delta(1-b)\left(e^{-\phi} + \phi e^{-\phi}\right)\right)} \tag{93}$$

$$A_v = \frac{e^{-\phi}(1-\underline{w})}{(1-\delta)\left(1 - \delta(1-b)\left(1 - e^{-\phi}\right)\right)} \tag{94}$$

$$A_v(1-\underline{w}) = \frac{\left(1 - \delta + \delta e^{-\phi}\right)(1-\underline{w})}{(1-\delta)\left(1 - \delta(1-b)\left(1 - e^{-\phi}\right)\right)} \tag{95}$$

Next we determine the two possible equilibrium wages. The higher wage $\overline{w}$, that comes about when several vacancies compete for a worker, must be such that all the vacancies are indifferent between paying the wage and continuing search for a worker, i.e.

$$1 - \overline{w} + \delta b A_v + \delta(1-b)A_v(1-\overline{w}) = \delta A_v. \tag{96}$$

Similarly, the lower wage $\underline{w}$, that comes about when a vacancy meets an unemployed alone and gets to make a take-it-or-leave-it offer, is such that the unemployed is indifferent between accepting the wage and continuing search, i.e.

$$\underline{w} + \delta b A_u + \delta(1-b)A_u(\underline{w}) = \delta A_u. \tag{97}$$

Using (92)-(95), and replacing $\overline{w}$ by $\underline{w}$ in (100), and $\underline{w}$ by $\overline{w}$ in (91), we can solve

$$\overline{w} = \frac{1-\delta\left(1-b\right)\left(e^{-\phi}+\phi e^{-\phi}\right)}{1-\delta(1-b)\phi e^{-\phi}} \tag{98}$$

$$\underline{w} = \frac{\delta(1-b)\left(1-e^{-\phi}-\phi e^{-\phi}\right)}{1-\delta(1-b)\phi e^{-\phi}} \tag{99}$$

Using these data we can finally solve for the expected utility of an unemployed worker who waits

$$A_u = \frac{1-e^{-\phi}-\phi e^{-\phi}}{\left(1-\delta\right)\left(1-\delta(1-b)\phi e^{-\phi}\right)} \tag{100}$$

and for the expected utility of a vacancy that moves

$$A_v = \frac{e^{-\phi}}{(1-\delta)(1-\delta(1-b)\phi e^{-\phi})} \tag{101}$$

### A3.1.2 Mixed strategy

The expected utility of an unemployed when the vacacies use a mixed strategy $H(w)$ with support $[l,h]$ is determined by

$$M_u = e^{-\phi}\delta M_u + \sum_{k=1}^{\infty} e^{-\phi}\frac{\phi^k}{k!}\int_l^h kh(w)H^{k-1}(w)M_u(w)dw. \tag{102}$$

Similarly, the utility of a worker who is employed at wage $w$ is given by

$$M_u(w) = w + \delta b M_u + \delta(1-b)M_u(w). \tag{103}$$

Solving $M_u(w)$ and inserting it back to (102) yields the following formula where the last two terms result from partial integration

$$\begin{aligned} M_u \quad = \quad & e^{-\phi}\delta M_u + \sum_{k=1}^{\infty} e^{-\phi}\frac{\phi^k}{k!} \,/_l^h\, H^k(w)(1-\delta+\delta b)^{-1}\delta b M_u + \\ & \sum_{k=1}^{\infty} e^{-\phi}\frac{\phi^k}{k!} \,/_l^h\, H^k(w)(1-\delta+\delta b)^{-1}w - \sum_{k=1}^{\infty} e^{-\phi}\frac{\phi^k}{k!}\int_l^h H^k(w)(1-\delta+\delta b)^{-1}dw. \end{aligned} \tag{104}$$

Finally, we can simplify this by doing the summations and by changing the order of the summation and integration in the last sum

$$M_u = (1-\delta)^{-1}(1-\delta(1-b)e^{-\phi})^{-1}\left\{h - le^{-\phi} - \int_l^h e^{-\phi(1-H(w))}dw\right\}. \tag{105}$$

The expected utility of a vacancy must be the same regardless of which element it chooses from the support of its mixed strategy. Let us determine the utility of a vacancy that uses $l$. It is determined by

$$M_v^{1-l} = e^{-\phi}\left[1 - l + \delta b M_v^{1-l} + \delta(1-b)M_v(1-l)\right] + (1-e^{-\phi})\delta M_v^{1-l}. \tag{106}$$

Analogously, if a vacancy uses $h$ its utility is determined by

$$M_v^{1-h} = 1 - h + \delta b M_v^{1-h} + \delta(1-b)M_v(1-h) \tag{107}$$

as offering the highest possible wage means that the wage is always accepted. Finally, if the vacancy offers wage $w$ its utility is determined by

$$M_v^{1-w} = \sum_{k=0}^{\infty} e^{-\phi}\frac{\phi^k}{k!}\left[\begin{array}{c} H^k(w)\left(1 - w + \delta b M_v^{1-w} + \delta(1-b)M_v(1-w)\right) \\ +\left(1 - H^k(w)\right)\delta M_v^{1-w} \end{array}\right] \tag{108}$$

The utilities of being in an employment relationship at a specific wage are easily determined from equations

$$M_v(1-l) = 1 - l + \delta b M_v^{1-l} + \delta(1-b)M_v(1-l) \tag{109}$$

$$M_v(1-h) = 1 - h + \delta b M_v^{1-h} + \delta(1-b)M_v(1-h) \tag{110}$$

$$M_v(1-w) = 1 - w + \delta b M_v^{1-w} + \delta(1-b)M_v(1-w) \tag{111}$$

Solving from these the expected utility and inserting in formulae (110)-(108) and in turn forcing them to yield the same expected utility as auction, namely that given by (94) allows us to solve for the endpoints of the support of the mixed strategy as well as the mixed strategy itself

$$l = \frac{\delta(1-b)\left(1 - e^{-\phi} - \phi e^{-\phi}\right)}{1 - \delta(1-b)\phi e^{-\phi}} \tag{112}$$

$$h = \frac{1 - \delta(1-b)\phi e^{-\phi} - e^{-\phi}}{1 - \delta(1-b)\phi e^{-\phi}} \tag{113}$$

$$H(w) = \frac{1}{\phi}\ln\frac{1 - \delta(1-b)}{(1-w)\left[1 - \delta(1-b)\phi e^{-\phi}\right] - \delta(1-b)e^{-\phi}} \tag{114}$$

We denote the cumulative distribution function for realised wages by $M(w)$ and the corresponding density function is denoted by $m(w)$.

$$\begin{aligned} M(w) &= \frac{e^{-\phi(1-H(w))} - e^{-\phi}}{1 - e^{-\phi}} && (115) \\ &= \frac{(1-\delta(1-b))e^{-\phi}}{(1-e^{-\phi})\left[(1-w)(1-\delta(1-b)\phi e^{-\phi}) - \delta(1-b)e^{-\phi}\right]} - \frac{e^{-\phi}}{1-e^{-\phi}}. && (116) \end{aligned}$$

The density function is

$$m(w) = \frac{\left(1-\delta(1-b)\phi e^{-\phi}\right)(1-\delta(1-b))e^{-\phi}}{(1-e^{-\phi})\left[(1-w)(1-\delta(1-b)\phi e^{-\phi})-\delta(1-b)e^{-\phi}\right]^2} \tag{117}$$

**A3.2 Workers send applications to firms**

**A3.2.1 Auction**

Let $\theta = \frac{u}{v}$. The utility of a vacancy is determined by

$$A_v = \left(e^{-\theta} + \theta e^{-\theta}\right)\delta A_v + \left(1 - e^{-\theta} - \theta e^{-\theta}\right) A_v(1-\underline{w}) \tag{118}$$

where $\underline{w}$ is the wage that a worker offers when there are two or more workers competing for a vacancy. Note that when a vacancy meets exactly one worker the worker makes a take-it-or-leave-it offer that leaves no surplus to the vacancy. The utility of an employer who employes at wage $\underline{w}$

$$A_v(1-\underline{w}) = 1 - \underline{w} + \delta b A_v + \delta(1-b)A_v(1-\underline{w}). \tag{119}$$

The expected utility of an unemployed worker is determined by

$$A_u = e^{-\theta} A_u\left(\overline{w}\right) + \left(1 - e^{-\theta}\right)\delta A_u \tag{120}$$

where $\overline{w}$ is the wage that a worker offers when it gets to make a take-it-or-leave-it offer. The utility of a matched worker who is paid wage $\overline{w}$

$$A_u(\overline{w}) = \overline{w} + \delta b A_u + \delta(1-b)A_u(\overline{w}). \tag{121}$$

From (118)-(121) we can solve

$$A_v = \frac{\left(1 - e^{-\theta} - \theta e^{-\theta}\right)(1-\underline{w})}{(1-\delta)\left(1-\delta(1-b)\left(e^{-\theta} + \theta e^{-\theta}\right)\right)} \tag{122}$$

$$A_v\left(1-\underline{w}\right) = \frac{\left(1 - \delta(e^{-\theta} + \theta e^{-\theta})\right)(1-\underline{w})}{(1-\delta)\left(1-\delta(1-b)\left(e^{-\theta} + \theta e^{-\theta}\right)\right)} \tag{123}$$

$$A_u = \frac{e^{-\theta}\overline{w}}{(1-\delta)\left(1-\delta(1-b)\left(1-e^{-\theta}\right)\right)} \tag{124}$$

$$A_u\left(\overline{w}\right) = \frac{\left(1-\delta+\delta e^{-\theta}\right)(1-\overline{w})}{(1-\delta)\left(1-\delta(1-b)\left(1-e^{-\theta}\right)\right)} \tag{125}$$

The two possible equilibrium wages are :

$\underline{w}$: Several workers compete for a vacancy, all unemployed are are indifferent between working with the wage and continuing search

$$\underline{w} + \delta b A_u + \delta (1 - b) A_u(\underline{w}) = \delta A_u. \tag{126}$$

$\overline{w}$: A worker is the only applicant and gets to make a take-or-leave-it offer. The firm is indifferent between accepting the wage and continuing search

$$1 - \overline{w} + \delta b A_v + \delta (1 - b) A_v(1 - \overline{w}) = \delta A_{v.} \tag{127}$$

Using (122)-(125), and replacing $\underline{w}$ by $\overline{w}$ in (119), and $\overline{w}$ by $\underline{w}$ in (121) we can solve

$$\overline{w} = \frac{1 - \delta (1 - b) (1 - e^{-\theta})}{1 - \delta(1 - b)\ \theta e^{-\theta}}, \tag{128}$$

$$\underline{w} = \frac{\delta (1 - b) e^{-\theta}}{1 - \delta(1 - b)\ \theta e^{-\theta}}. \tag{129}$$

Using (128) and (129) we can solve for the expected utility of a vacancy that waits

$$A_v = \frac{1 - e^{-\theta} - \ \theta e^{-\theta}}{(1 - \delta) (1 - \delta(1 - b)\ \theta e^{-\theta})}. \tag{130}$$

and for the expected utility of a moving unemployed worker

$$A_u = \frac{e^{-\theta}}{(1 - \delta)(1 - \delta(1 - b)\ \theta e^{-\theta})}. \tag{131}$$

**A3.2.2 Mixed strategy**

The expected utility of a vacancy when the workers use a mixed strategy $F(w)$ with support $[l, h]$ is determined by

$$M_v = e^{-\theta} \delta M_v + \sum_{k=1}^{\infty} e^{-\theta} \frac{\theta^k}{k!} \int_l^h k f(w) [1 - F(w)]^{k-1} M_v(w) dw. \tag{132}$$

The utility of a filled vacancy is

$$M_v(w) = 1 - w + \delta b M_v + \delta(1 - b) M_v(w). \tag{133}$$

Solving $M_v(w)$ and inserting it back to (132) yields the following formula where the last two terms result from partial integration

$$M_v = e^{-\theta} \delta M_v + \sum_{k=1}^{\infty} e^{-\theta} \frac{\theta^k}{k!} /_l^h \left[ - [1 - F(w)]^k \right] (1 - \delta + \delta b)^{-1} \delta b M_v + \tag{134}$$

$$\sum_{k=1}^{\infty} e^{-\theta} \frac{\theta^k}{k!} /_l^h \left[ - [1 - F(w)]^k \right] (1 - \delta + \delta b)^{-1} (1 - w) - \tag{135}$$

$$\sum_{k=1}^{\infty} e^{-\theta} \frac{\theta^k}{k!} \int_l^h [1 - F(w)]^k (1 - \delta + \delta b)^{-1} dw.$$

Finally, we can simplify this by doing the summations and by changing the order of the summation and integration in the last sum

$$M_v = (1-\delta)^{-1}(1-\delta(1-b)e^{-\theta})^{-1}\left\{1-l-(1-h)\,e^{-\phi}-\int_l^h e^{-\theta F(w)}dw\right\} \tag{136}$$

The expected utility of a worker must be the same regardless of which element he chooses from the support of its mixed strategy. The utility of a worker that uses $l$ is

$$M_u^l = l + \delta b M_u^l + \delta(1-b)M_u(l) \tag{137}$$

and the utility of a worker that uses $h$ is

$$M_u^h = e^{-\theta}\left[h + \delta b M_u^h + \delta(1-b)M_u(h)\right] + (1-e^{-\theta})\delta M_u^h \tag{138}$$

as offering the highest possible wage means that the wage is always accepted. Finally, if the vacancy offers wage $w$ its utility is determined by

$$M_u^w = \sum_{k=0}^{\infty} e^{-\theta}\frac{\theta^k}{k!}\left[[1-F(w)]^k\,(w+\delta b M_u^w + \delta(1-b)M_u(w)) + \left(1-(1-F(w))^k\right)\delta M_u^w\right] \tag{139}$$

The utilities of being employed at a specific wage are

$$M_u(l) = l + \delta b M_u^l + \delta\,(1-b)\,M_u(l) \tag{140}$$

$$M_u(h) = h + \delta b M_u^h + \delta\,(1-b)\,M_u(h) \tag{141}$$

$$M_u(w) = w + \delta b M_u^w + \delta\,(1-b)\,M_u(w) \tag{142}$$

Solving from these the expected utility and inserting in formulae (137)-(139) and in turn forcing them to yield the same expected utility as auction, namely that given by (131) allows us to solve for the endpoints of the support of the mixed strategy as well as the mixed strategy itself

$$l = \frac{e^{-\theta}}{1-\delta(1-b)\;\theta e^{-\theta}}, \tag{143}$$

$$h = \frac{1-\delta(1-b)\left(1-e^{-\theta}\right)}{1-\delta(1-b)\;\theta e^{-\theta}}, \tag{144}$$

$$F(w) = \frac{1}{\theta}\ln\frac{w\left[1-\delta(1-b)\;\theta e^{-\theta}\right]-\delta(1-b)e^{-\theta}}{e^{-\theta}(1-\delta(1-b))}. \tag{145}$$

We denote the cumulative distribution function for the realised wages by $G(w)$ and the corresponding density function is denoted by $g(w)$.

$$G(w) = \frac{1-e^{-\theta F(w)}}{1-e^{-\theta}} \tag{146}$$

$$= \frac{1}{1-e^{-\theta}} - \frac{(1-\delta(1-b))e^{-\theta}}{(1-e^{-\theta})\left[w(1-\delta(1-b)\,\theta e^{-\theta})-\delta(1-b)e^{-\theta}\right]} \tag{147}$$

The density function is

$$g(w) = \frac{\left(1-\delta(1-b)\,\theta e^{-\theta}\right)(1-\delta(1-b))e^{-\theta}}{(1-e^{-\theta})\left[w(1-\delta(1-b)\,\theta e^{-\theta})-\delta(1-b)e^{-\theta}\right]^2} \tag{148}$$

**A3.3 Equilibrium Market Structure**

In equilibrium, workers are indifferent between sending applications and receiving offers from firms, and firms are indifferent between making offers and receiving applications from workers. That is, for a worker

$$\frac{e^{-\theta}}{(1-\delta)(1-\delta(1-b)\theta e^{-\theta})} = \frac{1-e^{-\phi}-\phi e^{-\phi}}{(1-\delta)\left(1-\delta(1-b)\phi e^{-\phi}\right)}, \tag{149}$$

and for a firm

$$\frac{e^{-\phi}}{(1-\delta)(1-\delta(1-b)\phi e^{-\phi})} = \frac{1-e^{-\theta}-\theta e^{-\theta}}{(1-\delta)\left(1-\delta(1-b)\theta e^{-\theta}\right)}. \tag{150}$$

The left-hand side are utilities from sending wage demands or offers, and the right-hand sides are utilities from receving them. It turns out that in equilibrium $\theta = \phi \equiv \theta_0 \approx 1.146$, and there exists a unique equilibrium for strictly positive $x$ and $y$:

$$x = \frac{\theta_0(\alpha\theta_0-1)}{\alpha\left(\theta_0^2-1\right)}, \tag{151}$$

$$y = \frac{\alpha\theta_0-1}{\theta_0-1}, \tag{152}$$

where $\alpha = \frac{u}{v}$. However, two markets co-exist only if $x \in (0,1)$ and $y \in (0,1)$. These hold only if $\alpha \in \left(\frac{1}{\theta_0}, \theta_0\right)$. We see that two markets co-exist only if there are roughly equally many firms and workers in the economy.

Plugging the above solutions for $x$ and $y$ into $m(w)$ and $g(w)$ and using the appropriate ranges for the distributions just like in the two models above, we get the equilibrium distribution for the realised wages. The density function has approximately the same shape as the one produced by the static model.

### A3.4 Steady state

In a steady state equilibrium the number of matches equals the number of separations:

$$yv(1-e^{-\theta}) + (1-x)u(1-e^{-\phi}) = b(E-v), \tag{153}$$

where $yv(1-e^{-\theta})$ is the number of matches that form in a market where firms are urns, and $(1-x)u(1-e^{-\phi})$ is the number of matches in the market where workers are urns, and $b(E-v)$ is number of matches that break down per period. From (153) we can solve

$$v = \frac{Eb - (1-x)u(1-e^{-\phi})}{y(1-e^{-\theta}) + b} \tag{154}$$

On the other hand, from equation $E - v = W - u$ it follows that

$$v = E - W + u. \tag{155}$$

From (154) and (155) we can solve

$$u = \frac{y(1-e^{-\theta})(W-E) + Wb}{y(1-e^{-\theta}) + (1-x)(1-e^{-\phi}) + b} \tag{156}$$

$$v = \frac{(1-x)(1-e^{-\phi})(E-W) + Eb}{y(1-e^{-\theta}) + (1-x)(1-e^{-\phi}) + b} \tag{157}$$

When we substitute (151) and (152) for $x$ and $y$ in (156) and (157) we get that $u = f(W, E, b, \theta_0)$ and $v = g(W, E, b, \theta_0)$.

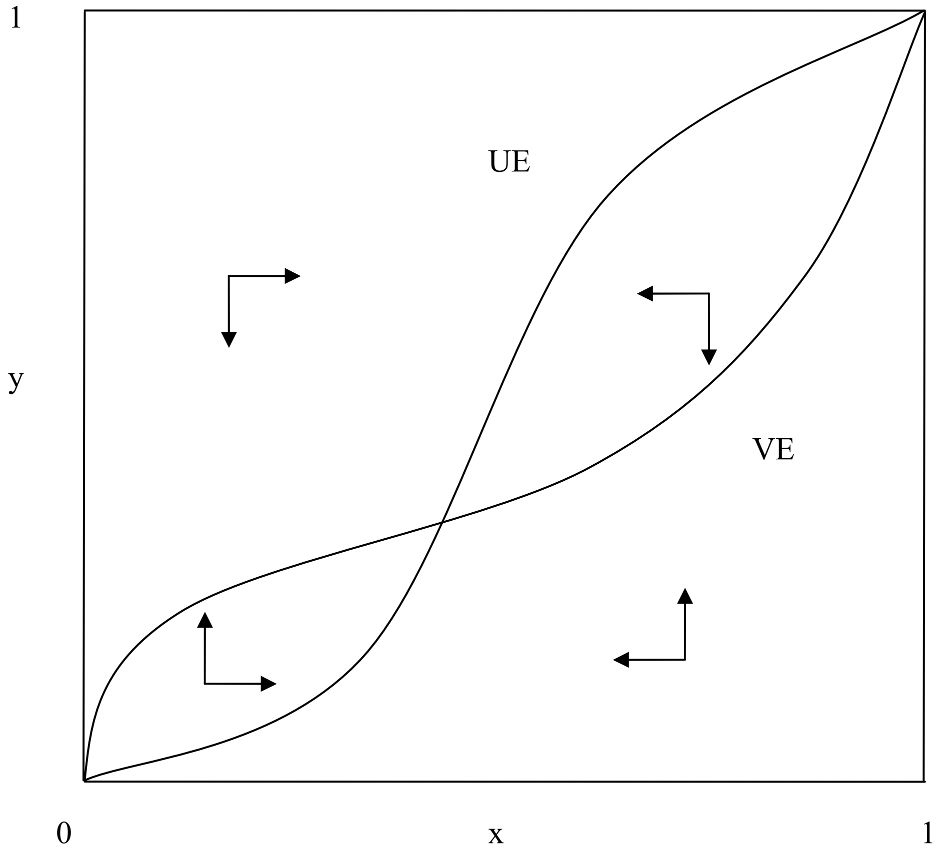


Figure 1: The mixed strategy equilibrium where $x \in (0,1)$ and $y \in (0,1)$ is stable.

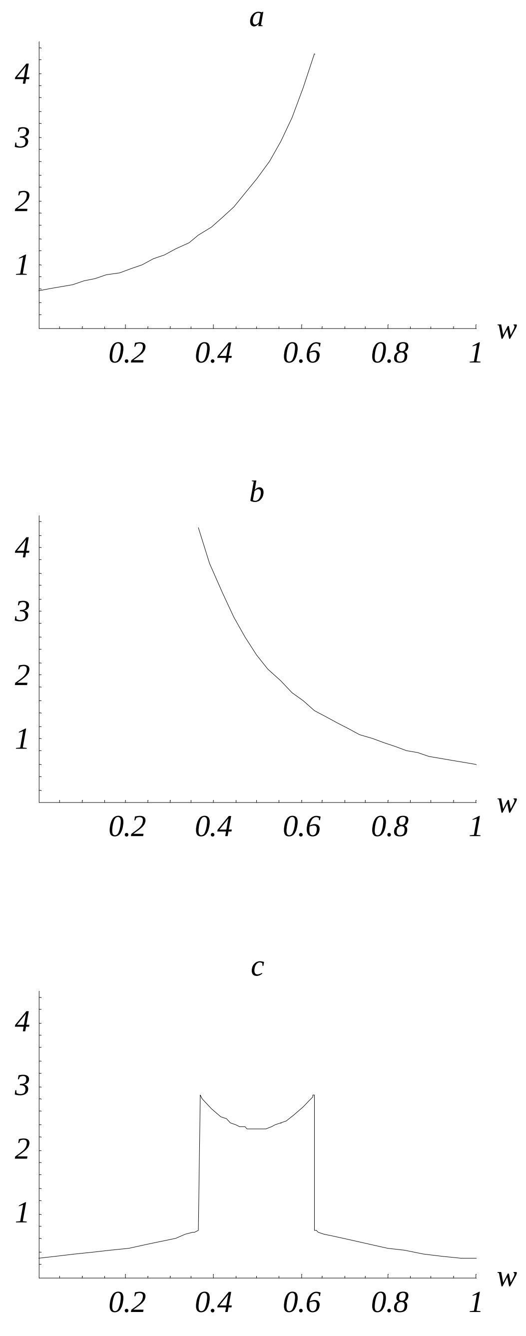


Figure 2: *Distributions of realized wages in the static model ( $u/v = 1$ ). (a) Market where vacancies send offers to job seekers. (b) Market where job seekers send offers to vacancies. (c) Two-sided market.*

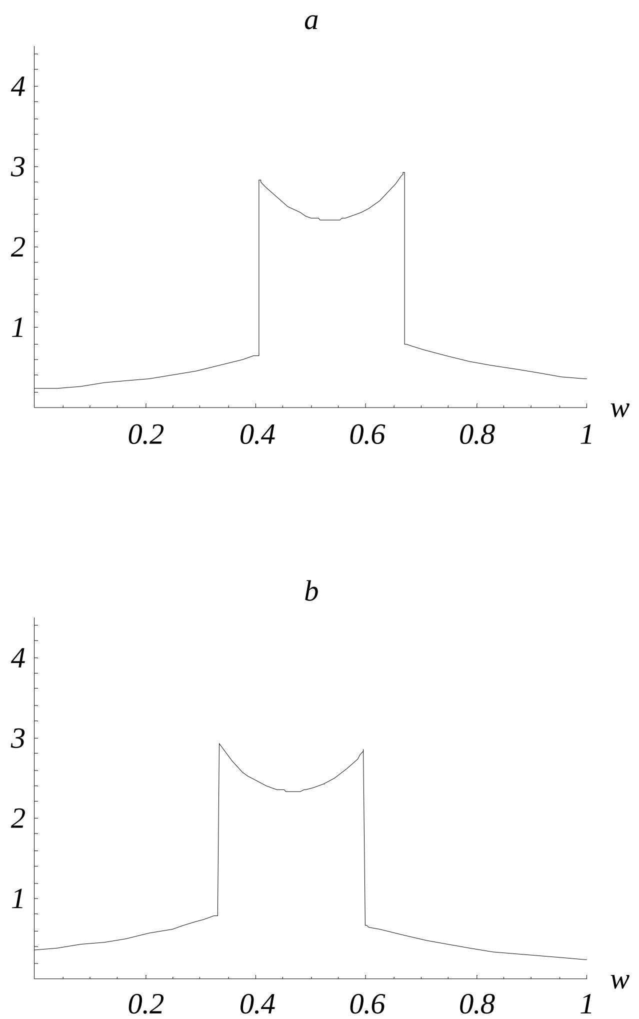


Figure 3: *Two-sided market in the static model, (a)* $u/v = 0.9$, *(b)* $u/v = 1.1$.