Investment Options and the Business Cycle

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Abstract

PRELIMINARY VERSION. A firm has investment options that it may use up immediately, or store for future use. A patent, e.g., is an option to implement an idea via a product or process innovation. Other investment options are protected by secrecy. An investment option is a profit opportunity that requires an investment to implement. Because investment options are scarce, Tobin's q is always above unity. When the stock of these options rises, the value of stock market falls, a result that exactly invalidates the use of the stock market as a positive indicator of the stock of intangibles. Finally, the stock market alone ensures that equilibrium is efficient.

1 Introduction

An investment option is a profit opportunity that requires an investment to implement. It is postponable if it is a patented invention, or if it is specific to a firm so that others cannot reduce its value by copying it. A firm has investment options that it may use up immediately, or store for future use. A patent, for instance, represents an investment option that only its holder can implement for a certain number of years. In a sense, even a trademark represents an option to produce a product that no one else can produce. Some investment options are protected not by law but by secrecy.

Investment options are a focus of the new Keynesian literature (Shleifer 1986), the strategic delay literature (Chamley and Gale 1994), and the investment literature (MacDonald and Siegel 1986, Dixit and Pindyck 1994, Abel and Eberly 2005). Several papers do discuss models of business cycles, e.g., Gale (1996), but do not try to fit data.

The model relates most closely to Yorukoglu (2000) and Abel and Eberly (2005) but studies different issues. It is a competitive GE model in which investment options, or "seeds" as I shall call them, are needed for the planting of trees. Seeds are produced

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by trees that are already planted, which I think of as the result of learning by doing. The number of trees grows over time and, in the absence of the seed constraint on investment, the model would be a standard one-sector Ak model with random TFP shocks. The main results are:

Intangibles and Q.—In the model, investment implements new ideas. Unimplemented ideas compete for capital, and when there are many around, their price falls, and with it so then does the price of claims to the output of existing ideas. In this sense, the more seeds we have on hand, the larger is the stock of what one would call intangibles, and the lower the value of our planted trees. This result is directly opposite to that of Hall (2000), who argued who used the value of stocks as a positive indicator of the stock of intangibles. Technically, the difference arises because Hall assumes variable proportions between tangible and intangible capital in production and no storage of intangibles, whereas I assume the opposite: Fixed proportions in production and storage. Measures of intangibles based on patent applications and trademarks co-move negatively with Tobin's Q, thus supporting my model.

Investment and Q.—Because seeds are scarce, the value of planted trees and thus Tobin's Q, is always above unity. The standard model with convex adjustment costs also has Q above unity, but that model forces a smoothness on investment that the present model does not. Instead, the model induces an intertemporal substitutability on investment. Thus, at least when agents have perfect foresight about TFP shocks, the model generates more volatile investment than the standard convex-adjustment model, and for roughly the same reason that the Rogerson-Hansen economy has more volatile employment than does the standard RBC model, at least when there is perfect foresight about shocks to wages. I expect the result to still hold when TFP shocks are persistent enough.

Decentralization and empirics.—These results hold in the planner's optimum which has two decentralizations. The first is a complete-markets decentralization in which a market for seeds exists. The second decentralization has no seeds market, only a market for shares of firms. The outcome for quantities and prices remains the same. It remains to be seen whether the efficiency of the no-seeds decentralization survives when firms differ. Jovanovic and Braguinsky (2004) develop a related one-period model in which firms differ in how many seeds they have and in the eventual productivity of trees that they may get to plant; they find that even without a seed market, takeovers (which are still transactions in the market for firms only) achieve efficiency. The results here are consistent with theirs.

Section 2 presents the model, section 3 describes a complete-markets decentralization, section 4 an incomplete-markets one. Section 5 reports simulations, compares the model to the data, and briefly describes some possible extensions. Several proofs and two deterministic versions of the model – one in discrete and one in continuous time – are reported in the Appendix.

2 Model

The model is that of a growing economy with two types of capital – trees, k, and seeds, S. A seed represents an option, storable indefinitely, to plant exactly one tree.

Production of output.—Output of fruit is

$$Y = zk.$$

If X is the number of trees newly planted, k evolves as

$$k' = k + X. \tag{1}$$

Production of seeds.—Let S denote the stock of seeds. New seeds are produced by existing trees. Each period a tree gives rise to λ new seeds, i.e., a total of

$$\text{new seeds} = \lambda k \tag{2}$$

Thus seeds grow via a process like learning by doing that takes up no resources.

The planting of trees.—Planting a tree requires a unit of fruit *and* a seed. Only one tree per seed can be planted, after which the seed is used up. Let S be the stock of seeds and let X be the number of trees planted. Then S evolves as

$$S' = \lambda k + S - X. \tag{3}$$

Since X is subtracted from the stock, a seed can be used to plant exactly one tree. Thus investment is Leontieff in two inputs, seeds and fruit. Their proportions are equal, an assumption that we shall drop when we get to the empirics, along with the assumption that neither k nor S depreciate. Leontieff investment implies that output too is Leontieff in seeds and fruit. Seeds are storable whereas fruit is not.

Timing.—Investment, X, is chosen *after* the trees produce zk units of fruit and after λk new seeds. Since $S' \geq 0$, the constraint on X is

$$X \leq \lambda k + S. \tag{4}$$

Thus investment is Leontieff in two inputs: seeds and fruit. We shall let investment be reversible: [1]

The income identity.—The cost of planting a tree is, as usual, one unit of fruit. Letting C be the consumption of fruit, the income identity is

$$zk = C + X. \tag{5}$$

[1] Unlike Sargent (1980), we shall not impose the constraint $X \geq 0$. This constraint is never violated in any of the simulations which assume that $\sigma = 2$. With a much lower value of σ and/or with a very persistent and variable z_t process, X would at times be negative.

The shocks.—We assume that the shocks follow the first-order Markov process: $\Pr(z_{t+1} \leq z' \mid z_t = z) = F(z', z)$, and that z' is stochastically increasing in z.

Preferences.—For $\sigma > 0$ and $\beta < 1$, preferences are

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \right\}.$$

The standard one-sector growth model.—It arises when the inequality in (4) never binds. The latter occurs when λ is large enough, e.g., if λ exceeds the largest possible z. It also occurs, *de facto*, when the initial stock of seeds, S_0, is so large that (4) does not come into play for a very long time.

2.1 The planner's problem

The state is (k, S, z), and the decision, x, is constrained by (42). The Bellman eq. is

$$v(k, S, z) = \max_{X \leq \lambda k + S} \left\{ \frac{(zk - X)^{1-\sigma}}{1-\sigma} + \beta \int v(k + X, \lambda k + S - X, z')\, dF \right\}. \tag{6}$$

Lemma 1 *A unique solution v to (6) exists, and is is strictly concave in (k, S). Moreover, X is increasing in S and, if z is i.i.d., in z.*

Proof. (i) Existence, uniqueness: Let T denote the operator on the RHS of (6). The operator is a contraction and maps continuous functions v into continuous and **unbounded** functions (Tv). Methods of Alvarez and Stokey (2000) show that....(ii) Concavity: We shall show that if $\tilde{v}$ is concave then $T\tilde{v}$ is strictly concave. Let $0 \leq \alpha \leq 1$. The constraint (4) is convex and its boundary is linear in S and k. Therefore if X_1 is feasible and optimal for the state (k_1, S_1) and X_2 is feasible and optimal for (k_2, S_2), then $X_\alpha \equiv \alpha X_1 + (1-\alpha) X_2$ is feasible, though not necessarily optimal for $(\alpha k_1 + (1-\alpha) k_2, \alpha S_1 + (1-\alpha) S_2)$. Therefore if $0 < \alpha < 1$

$$\begin{aligned} T\tilde{v}(\alpha k_1 + (1-\alpha) k_2, \alpha S_1 + (1-\alpha) S_2) &\geq \frac{(zk - X_\alpha)^{1-\sigma}}{1-\sigma} + \beta \int \tilde{v}(k + X_\alpha, \lambda k + S - X_\alpha, z')\, dF \\ &> \alpha T\tilde{v}(k_1, S_1) + (1-\alpha) T\tilde{v}(k_2, S_2) \end{aligned}$$

Therefore the operator transforms weakly into strictly concave functions. Therefore, the operator being a contraction, its unique fixed point v is strictly concave. (iii) Properties of X: (Here I assume the differentiability needed. Later, first derivatives of v will be shown to exist independently of the results of this Proposition). The FOC is

$$\xi(X, S) \equiv -(zk - X)^{-\sigma} + \beta \int \frac{dv}{dX} v(k + X, \lambda k + S - X, z')\, dF = 0$$

We have dropped k and z from the arguments of ξ as they play no role in the argument to be made. We now argue in 3 steps: (A) If a function of one variable H is twice differentiable with $H'' < 0$, then

$$\frac{\partial}{\partial S}\left(\frac{\partial}{\partial X}H\left[\lambda k + S - X\right]\right) = -H''(\cdot) > 0$$

Therefore, concavity of v in S alone implies $\frac{\partial}{\partial S}\frac{dv}{dX}\left(= -\frac{\partial^2 v}{\partial S^2}\right) > 0$; earlier, under (ii) we showed that concavity of v in (k, S) implies concavity of v in X holding (k, S) fixed – i.e., that $\frac{d^2 v}{dX^2} < 0$ and (B) Therefore $\xi_X < 0$ and $\xi_S > 0$. And, when z is i.i.d., $\xi_z =$ (C) Therefore $\frac{\partial X}{\partial S} = -\frac{\xi_S}{\xi_X} > 0$. ■

Reducing the state space.—From (43), $s' = \frac{\lambda + s - x}{1+x}$. The following result allows us to reduce the state space to just (s, z):

Lemma 2 *For $\sigma \neq 1, v$ is of the form*

$$v(k, S, z) = w(s, z)\, k^{1-\sigma},$$

where $w(s, z) = v(1, s, z)$, and where w satisfies

$$w(s, z) = \max_x \left\{ \frac{(z - x)^{1-\sigma}}{1-\sigma} + (1 + x)^{1-\sigma}\beta \int w\left(\frac{\lambda + s - x}{1 + x}, z'\right) dF \right\} \tag{7}$$

subject to (42). Moreover, v and w are of the same sign as $1 - \sigma$.

The proof (not reported) substitutes the desired functional form for v on the RHS of (6), and verifies that the same functional form emerges on the LHS. The case $\sigma = 1$ is covered separately below. Similar results are in Alvarez and Stokey (2000).

Corollary 1 *A unique solution w to (7) exists that is increasing and concave in s.*

Proof. *Existence*: Since a unique v exists, $w(s, z) = v(k, S, z)\, k^{-(1-\sigma)}$ is the unique solution for w. *Increasing*: In (42), a rise in s relaxes the constraint on x. Moreover, if one inserts on the RHS of (7) a function w that increases in s, evidently the property is preserved. *Concave*: The concavity of $v(k, S, z)\, k^{-(1-\sigma)}$ in S for fixed k implies that w is concave in s. ■

Corollary 2 *The policy $x(s, z)$ is increasing in s and, if z is i.i.d., increasing in z.*

Proof. All changes in $s \equiv S/k$ can be interpreted as changes in S for a given k. By Lemma 2, X is, for all k, increasing in S. For fixed k, a rise in S implies a rise in s and in x. The claim about z follows at once from Lemma 2. ■

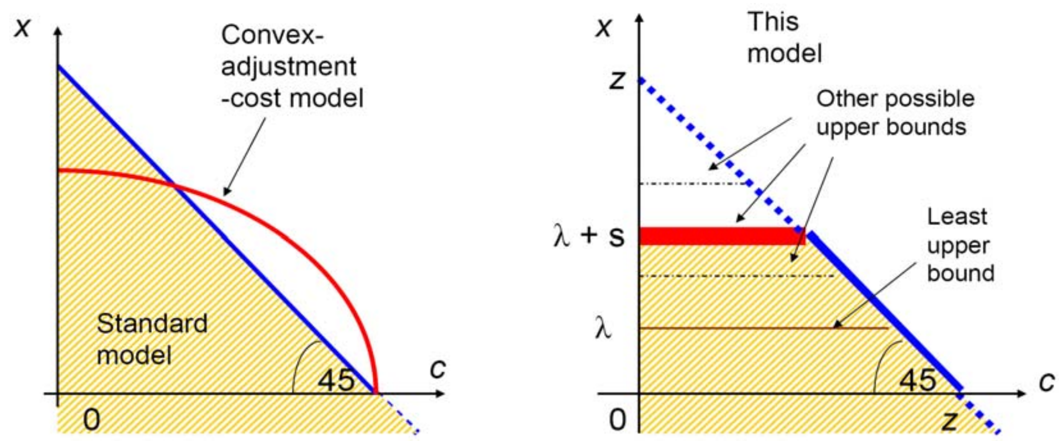

Figure 1: Relation to the convex adjustment-cost model

The relation to other models is easily seen graphically. In its left panel, Figure 1 shows the consumption-investment trade-off in the standard model and the convex-adjustment-cost model. In its right panel,the Figure shows the constraint imposed by a particular upper bound on x, namely $\lambda + s$. Since $s \geq 0$, investment can never be constrained by any number smaller than λ, and so that's the tightest constraint on x that can possibly arise. The position of the constraint will depend on what has been happening earlier. In particular, an "seed crunch" and with it a high value of Q will turn out to be more likely following a prolonged boom caused by a succession of large realizations of z. Such realizations are likely to draw s to its minimum level of zero, leading the constraint to be at λ.

We can also illustrate in terms of the marginal cost of investment. Let

$$C(x, s) = \frac{\text{investment cost}}{\text{capital stock}} = \begin{cases} x & \text{if } x \leq \lambda + s \\ \infty & \text{otherwise} \end{cases}$$

denote the cost of investment, in units of fruit. The marginal adjustment costs, $\frac{\partial}{\partial x} C(x, s)$, are drawn in Figure 2. Other microfoundations – time to build – is also related, but more complicated. If time to build is T periods, then there are, in principle, T capital stocks, the capital that is productive now, and $T - 1$ capital types, indexed by the number of periods' waiting time until it becomes productive.

In sum there are two differences between this model and the standard one. First, the shape of the feasible set is different, as Figure 1 shows. And, second, there is intertemporal substitution in investment.

Lemma 3 *w is strictly increasing in z.*

Proof. Since $x \geq -1$ and since z' is stochastically increasing in z, for any function $w(s, z)$ increasing in z', the second term on the RHS of (7) is increasing in

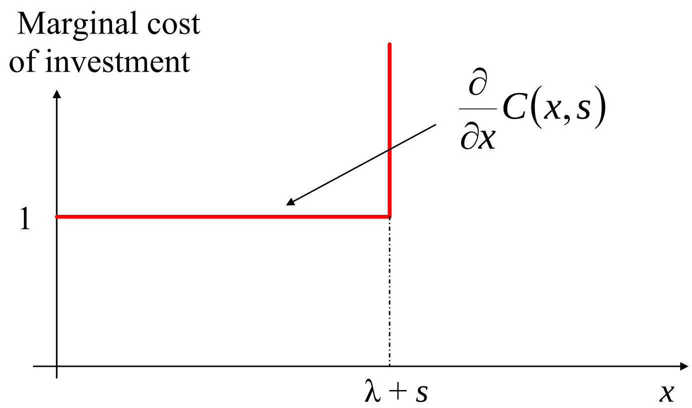

Figure 2: Marginal adjustment costs

z. Moreover, since $C \geq 0$, the first term on the RHS of (7) is strictly increasing in z.
■

Lemma 4 *w is differentiable with respect to s, with derivative*

$$w_s = \frac{1}{1+\lambda+s}\left([1-\sigma]\,w - (1+z)\,[z-x]^{-\sigma}\right) > 0 \tag{8}$$

for all (s, z).

The proof is in Appendix 1; if follows the proof of proposition 2 of Lucas (1978) but is complicated by the seed constraint.

Note that the term $(1-\sigma)\,w$ is positive for all $\sigma \neq 1$ because for $\sigma > 1$, $w < 0$.

Lemma 5 *The optimal policy $x\,(s,z)$satisfies*

$$1 - \beta \int \left(\frac{(1+x)}{z-x}\right)^{-\sigma} \left[(z'-x')^{-\sigma}\,(1+z') + \lambda w'_s\right] dF \begin{cases} = 0 & \text{if } s' > 0 \\ \leq 0 & \text{if } s' = 0 \end{cases}. \tag{9}$$

Proof. By Lemma 2, v is differentiable w.r.t. k, and if w is differentiable *w.r.t.* s, so is v w.r.t. S. Then the FOC is

$$C^{-\sigma} - \beta \int (v_k - v_S)\,dF \leq 0, \tag{10}$$

with equality if $S' > 0$. We have

$$v\,(k,S,z) = \max_{S'} \left\{ \frac{(zk+S'-\lambda k-S)^{1-\sigma}}{1-\sigma} + \beta \int v\,(k+\lambda k+S-S', S', z')\,dF \right\}.$$

The envelope result (since S does not enter the constraint $S' \geq 0$) is

$$v_S = -C^{-\sigma} + \beta \int v_k dF$$

and

$$v_k = (z-\lambda) C^{-\sigma} + (1+\lambda) \beta \int v_k dF = (z-\lambda) C^{-\sigma} + (1+\lambda) \left(v_S + C^{-\sigma} \right) = (1+\lambda) v_S + (1+z) C^{-\sigma} \tag{11}$$

But by Lemma 2, $v(k, S, z) = w\left(\frac{S}{k}, z\right) k^{1-\sigma}$ so that

$$v_k = (1-\sigma) w k^{-\sigma} - s w_s k^{-\sigma} \quad \text{and} \quad v_S = w_s k^{-\sigma}$$

Now, from (11), $v_k = (1+\lambda) v_S + (1+z) C^{-\sigma}$, so that the FOC becomes

$$C^{-\sigma} - \beta \int \left(\lambda v'_S + (1+z) C'^{-\sigma} \right) dF \leq 0$$

But $v_S = w_s k^{-\sigma}$ and the above equation then reads

$$\begin{aligned} 0 &\geq (z-x)^{-\sigma} k^{-\sigma} - \beta \int \left(\lambda w'_s (k')^{-\sigma} + (1+z')(z'-x')^{-\sigma} (k')^{-\sigma} \right) dF \\ &= \left(\frac{z-x}{1+x} \right)^{-\sigma} - \beta \int \left(\lambda w'_s + (1+z')(z'-x')^{-\sigma} \right) dF, \end{aligned} \tag{12}$$

from which (9) follows. ■

2.1.1 The set on which (4) binds

Consumption is most volatile and investment least volatile when (4) binds. Let $\Delta = \{(s,z) \mid x(s,z) = \lambda + s\}$ be the set of states for which (4) binds. In this region, X cannot respond to z and therefore C moves one-for-one with zk and, hence, is more volatile than in the standard model. True, this statement is conditional on s, but for $(s,z) \in \Delta$, $s' = 0$, and $x' = x(0, z')$. If (s,z) remain in Δ for more than one period, then in period two and beyond,

$$x(0,z) = \lambda \text{ and } c = z - \lambda.$$

The further z is from being a random walk (and it seems to depart substantially from it, see (37)) below, the more these rules depart from what the standard model would predict. In contrast, when $s \to \infty$, we get the standard model, for then the probability that (4) will bind in the foreseeable future goes to zero. Because x is increasing in z, Δ contains large z values. For $(s,z) \in \Delta$, $s' = 0$ so that $x' = x(0, z')$. Let $z^*(s) = \inf_{\{z \mid (z,s) \in \Delta\}} z$ be the boundary of Δ. Then, as Figure 3 illustrates, we can then show the following:

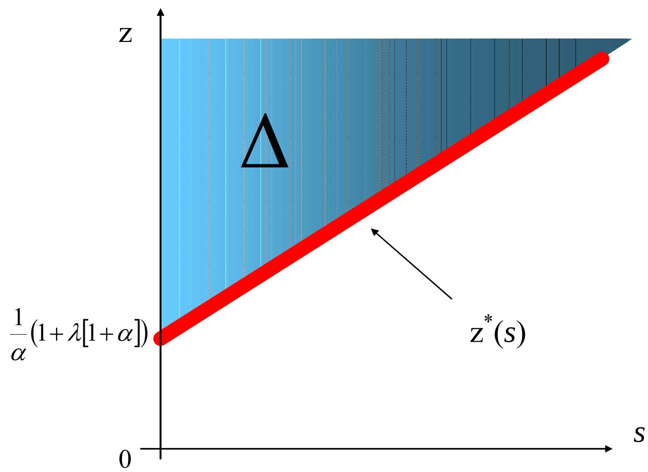

Figure 3: The set Δ when z is i.i.d.

Proposition 1 *If $z \sim F(z)$ is i.i.d., then*

$$z^*(s) = \frac{1 + (1+\alpha)(\lambda + s)}{\alpha}, \tag{13}$$

where α is the constant:

$$\alpha = \left(\beta \int \frac{\lambda (1-\sigma) w(0, z') - (1+z')(z' - x[0, z'])^{-\sigma}}{1+\lambda} dF(z') \right)^{1/\sigma}. \tag{14}$$

Proof. From (9) and from an updated version of (8) we have

$$\beta \int \left(\frac{1+x}{z-x} \right)^{-\sigma} \frac{\lambda (1-\sigma) w' - (1+s')(1+z')(z'-x')^{-\sigma}}{1+\lambda+s'} dF \begin{cases} = 1 & \text{if } s' > 0 \\ \geq 1 & \text{if } s' = 0 \end{cases},$$

i.e.,

$$\beta \int \frac{\lambda (1-\sigma) w' - (1+s')(1+z')(z'-x')^{-\sigma}}{1+\lambda+s'} \begin{cases} = \left(\frac{1+x}{z-x}\right)^{\sigma} & \text{if } s' > 0 \\ \geq \left(\frac{1+x}{z-x}\right)^{\sigma} & \text{if } s' = 0 \end{cases}$$

i.e.,

$$\alpha \begin{cases} = \frac{1+x}{z-x} & \text{if } s' > 0 \\ \geq \frac{1+x}{z-x} & \text{if } s' = 0 \end{cases}$$

On the other hand, if x is constrained and held constant at $\lambda + s$ as z varies, the RHS is decreasing in z. Large z's make the inequality strict. We find the smallest one that will allow strict equality at $x = \lambda + (1+\lambda) s$. Setting it at equality we have $1 + \lambda + s = \alpha (z - \lambda - s)$, i.e., (13). Moreover, for $z = z^*(s)$ at $s' = 0$ so that $x' = x(0, z')$, and $w' = w(0, z')$, which yields (14). ■

Even when z is i.i.d., it still raises x because a higher z today raises wealth and causes a rise in desired future consumption.

When z is serially correlated, the boundary of Δ is no longer linear but Δ retains a shape similar to the one portrayed in Figure 3: $z^*(s)$ still solves (13) in which α is replaced by

$$\alpha(z) = \left(\beta \int \frac{\lambda(1-\sigma)\, w(0,z') - (1+s')(1+z')(z' - x[0,z'])^{-\sigma}}{1+\lambda+s'} dF(z',z)\right)^{1/\sigma}.$$

While x is less volatile on Δ, to achieve a given growth rate, x must make up for its low mean on Δ with a higher mean off of Δ, which introduces a force towards bimodality in the distribution of x and a higher volatility of x.

3 Complete markets

Assume that a market for seeds exists. A case can be made that such a market indeed does indeed exist. Takeovers play a part in achieving transfers of what we call seeds. A firm can be said to buy seeds when it acquires a company for its intellectual property; this is a fairly thick market in which Microsoft and Pfizer, e.g., have been highly active. Firms also buy patent rights from individuals directly and from one another. A firm can be said to sell seeds when it spins off some activity, or when it hires people at wages that include a negative compensating differential for the value that its workers will draws from the experience gained; such a market is modeled, e.g., by Chari and Hopenhayn (1991). An example of employees walking out with seeds is Xerox in the 70's – it had inventions that it was unable or unwilling to implement and that were later marketed by its former employees.

Let $p(s,z)$ be the price of seeds, and $q(s,z)$ the price of a planted tree without a claim on its current-period dividends. A firm pays all its net income in dividends every period. All trade in seeds is between firms.

Firms.—A firm consists of the trees it has planted and of seeds it has stored. The firm maximizes its value. That is, it solves

$$Pk = \max_{X,Y'} \{zk - X + (k+X)\, q + pS'\}$$

subject to (3) but not (4); the firm can support any level of investment X by a seed purchase, so that S' can be negative. Of course, (4) will have to hold in the aggregate. Substituting from (3) for S', the firm's problem becomes

$$\max_X \{(z+q)\, k + p(S+\lambda k) + (q - [1+p])\, X\}$$

Arbitrage.—If q differed from $1+p$ the firm could drive dividends to plus infinity by sending X to plus or to minus infinity. A negative X would entail selling off k and the seeds that it embodies at a price of $1+p$ and paying the net proceeds out as dividends.[2] These extreme outcomes cannot arise in equilibrium, we must have the "no-arbitrage condition"[3]

$$q = 1 + p. \tag{15}$$

which, when substituted into the maximand, means that the firm's cum-dividend value is

$$P = (z + q)\, k + p\,(S + \lambda k)\,. \tag{16}$$

We shall obtain q from the household's problem, and then (15) gives us p.

Households.—Let $k = \#$ of trees owned by the household. The household's budget constraint therefore is

$$qk' + C = zk + qk. \tag{17}$$

The RHS of (17) gives the household's dividend receipts which are proportional to total resources, the LHS describes how they are spent.

The household's Bellman eq.—The household's personal state is the pair (k, S), and it takes (s, z) and their laws of motion as given. Its Bellman equation is

$$V(k, s, z) = \max_{k' \geq 0} \left\{ \frac{(zk - q(s, z)\,[k' - k])^{1-\sigma}}{1-\sigma} + \beta \int V(k', S', s'(s, z), z')\, dF \right\} \tag{18}$$

with $q(s, z)$ and $s'(s, z)$ taken as given.

Since the household gets all the rents, optimality of the equilibrium occurs if and only if $v = V$. For p to equal its marginal social value in consumption units, we should have $p = \frac{v_S}{C^{-\sigma}}$

Proposition 2 *Optimum and equilibrium coincide; for all states,*

$$v = V \qquad \textit{and} \quad p = \frac{v_S}{C^{-\sigma}}.$$

Proof. The FOC is

$$-C^{-\sigma} q + \beta \int V_k' dF = 0. \tag{19}$$

[2] The most relevant real-life counterpart of this is when a company sells off a division, or when it is acquired.

[3] This arbitrage condition would hold even if we imposed the constraint that aggregate investment be nonnegative. An individual firm could have $X < 0$ without affecting aggregatesOn the other hand, if the salvage value of k were less than unity, (15) would read

$$q \leq 1 + p \qquad \text{when } X \leq 0.$$

If $v = V$, (19) reads $-C^{-\sigma} - v_S + \beta \int v'_k dF = 0$, which implies that

$$v_S = -C^{-\sigma} + \beta \int v'_k dF \tag{20}$$

But (6) can be written as

$$v(k, S, z) = \max_{S' \geq 0} \left\{ \frac{(zk - [\lambda k + S - S'])^{1-\sigma}}{1-\sigma} + \beta \int v(k + \lambda k + S - S', S', z') \, dF \right\}, \tag{21}$$

and differentiating w.r.t. S, (20) follows. This implies that the household's choice of k' should coincide with that of the planner. ■

Finally, let us show that P, the value of the firm, equals the marginal social value, in consumption units, of the capital that it contains.

Corollary 3 *The value of firms equals the marginal social value of the* (k, S) *bundle that they contain:*

$$P = \frac{v_k + s v_s}{C^{-\sigma}} = \frac{(1-\sigma) w}{c^{-\sigma}}. \tag{22}$$

Proof. The first equality in (22): Since we have established that $p = \frac{v_S}{C^{-\sigma}}$, we need only show that

$$z + q + p\lambda = \frac{v_k}{C^{-\sigma}}$$

But from (21),

$$\frac{v_k}{C^{-\sigma}} = z - \lambda + \frac{(1+\lambda)\beta}{C^{-\sigma}} \int v'_k dF,$$

and therefore we need to show that

$$q + p\lambda = -\lambda + \frac{(1+\lambda)\beta}{C^{-\sigma}} \int v'_k dF.$$

Now, since $q + p\lambda + \lambda = q(1+\lambda)$, we simply need to show that

$$q = \frac{\beta}{C^{-\sigma}} \int v'_k dF$$

but this follows from (19). **The second equality in** (22): Since $v = w\left(\frac{S}{k}, z\right) k^{1-\sigma}$,

$$v_k = k^{-\sigma}\left([1-\sigma] w - s w_s\right) \quad \text{and} \quad v_s = w_s k^{-\sigma}$$

and the second equality follows. ■

Calculating q and p.—Optimum and equilibrium are the same, and therefore p must equal the marginal social value of a seed:

$$p(s, z) = \frac{1}{U'(C)} v_S = (z - x)^{\sigma} w_s(s, z).$$

because $\frac{1}{U'(C)} = \frac{(z-x)^{\sigma}}{k^{-\sigma}}$ and $v_S = \frac{1}{k} w_s(s, z) k^{1-\sigma}$.

Proposition 3 *$p(s,z)$ is decreasing in s*

Proof. By Corollary 1, w is concave in s which means that w_s is decreasing in s. By Corollary 2, x is increasing in s so that $(z-x)^\sigma$ is decreasing in s. ■

Using (8),

$$\begin{aligned} p(s,z) &= (z-x)^\sigma \frac{1}{1+\lambda+s}\left([1-\sigma]w-(1+z)[z-x]^{-\sigma}\right) \\ &= \frac{1}{1+\lambda+s}\left(\frac{1-\sigma}{(z-x)^{-\sigma}}w-[1+z]\right) \end{aligned} \tag{23}$$

Finally, (15) gives us q.[4]

3.0.2 Measurement of Tobin's Q

Since s is probably not in the firm's book value, by Q or "Tobin's Q" we shall mean the firm's ex-dividend value per unit of k. That is, if D is the firm's dividend, then

$$Q = \frac{P}{k} - \frac{D}{k} \tag{24}$$

We shall now see that if we include capital gains as part of dividends (which is in any case needed if dividend policy is to be neutral in its effect on Q), then $Q = q$.

If the firm were to hold no seeds into the next period but, instead, sell them and pay out the proceeds in dividends along with its net earnings, its dividends per unit of k would equal

$$\frac{\hat{D}}{k} = z - x(1+p) + p(s+\lambda).$$

In addition to $\hat{D}$, however, the owners of the firm also enjoy capital gains, the expected value of which is just the current value of the newly-planted trees, i.e., qX. Therefore dividends plus capital gains are

$$\frac{D}{k} = \frac{\hat{D}}{k} + qx = z + p(s+\lambda)$$

Substituting into (24) yields and using (16) yields

$$\begin{aligned} Q &= z + q + p(s+\lambda) - [z - x(1+p) + p(s+\lambda)] - qx \\ &= q. \end{aligned}$$

[4] As a check, the Appendix works out the case case of a large λ or s_0 in the decentralization with seeds. For large values of λ, the seed constraint would either never bind and seeds would not have any value, and for large s_0, the seed constraint would not bind for so many periods into the future that it would have no practical relevance for policies or prices. Either way, we should have $q \to 1$. The Appendix verifies that this is indeed so.

4 Incomplete markets

Suppose now that firms' shares trade but that seeds do not. In this case, all seeds have to be stored by the firms that produced them, and the representative firm holds both trees and seeds under its roof, i.e., it is the tree-seed bundle (k, S). The only asset that a household can own is a claim on the dividends paid by such a firm. Therefore this decentralization has just two markets: A market for output, and a market for firm's shares. Since the number of date-t goods (consumption, capital, and seeds) is three, the number of goods exceeds the number of markets, and we do not expect recursive competitive equilibrium to be optimal. What follows extends the recursive equilibrium of Mehra and Prescott (1980) to a growing production economy, as done in Jovanovic (2006, Sec. 4).

Assume a continuum of firms of measure one and an equal number of shareholders. Equilibrium then requires that each shareholder hold exactly one share. Shareholders have no other income. They take as given firms' policies $x(s, z)$. Firms pay $(z - x[s, z])\,k$ dividends in state (k, s, z).

The savings decision.—With n shares, a shareholder's wealth is the current dividend, $(z - x)\,k$ plus the value of his holdings, $\hat{Q}[s, z]\,kn$. This wealth is spent on consumption and on future holdings of shares $\hat{Q}(s, z)\,kn'$. Thus $\hat{Q}kn' + C = \left([z - x]\,k + \hat{Q}k\right) n$, or after dividing through by k,

$$\hat{Q}n' + c = \left(z - x + \hat{Q}\right) n,$$

so that

$$c = (z - x)\,n + \hat{Q}\,(n - n')$$

where x is given to the shareholder. The shareholder takes the aggregate law of motion of $k'(s, z)$ $x(s, z)$ and $s'(s, z)$ as given. His state is (k, n, s, z), and, with some of the arguments (s, z) dropped from the notation, his Bellman equation then is

$$V(k, n, s, z) = \max_{n'} \left\{ \frac{\left([z - x]\,n + \hat{Q}\,[n - n']\right)^{1-\sigma} k^{1-\sigma}}{1 - \sigma} + \beta \int V(k', n', s', z')\,dF \right\}.$$

Again, V will be homogeneous of degree $1-\sigma$ in k, with $V(k, n, s, z) = W(n, s, z)\,k^{1-\sigma}$ and so we eliminate k to get

$$W(n, s, z) = \max_{n'} \left\{ \frac{\left([z - x(s, z)]\,n + \hat{Q}(s, z)\,[n - n']\right)^{1-\sigma}}{1 - \sigma} + \beta\,(1 + x[s, z])^{1-\sigma} \int W(n', s'[s, z], z')\,dF \right. \tag{25}$$

Equilibrium requires that $n'(1,s,z) = 1$. Evaluated at equilibrium, the first-order condition is

$$(z - x[s,z])^{-\sigma} \hat{Q}(s,z) = \beta (1 + x[s,z])^{1-\sigma} \int W_n dF. \tag{26}$$

The consumer faces no constraints on n' other than the budget constraint. The envelope theorem can be applied as we shall only ask that the derivative exist at $n' = 1$. That derivative is

$$W_n(1,s,z) = (z - x[s,z])^{-\sigma} \left[z - x(s,z) + \hat{Q}(s,z) \right].$$

When updating this, it must be multiplied by $k'/k = 1 + x$. Upon an update and a substitution into (26) gives the revised FOC and pricing formula

$$\begin{aligned} \hat{Q}(s,z) &= \beta (1 + x[s,z])^{1-\sigma} \int \frac{(z' - x[s'(s,z), z'])^{-\sigma}}{(z - x[s,z])^{-\sigma}} \left(z' - x[s'(s,z), z'] + \hat{Q}(s'[s,z], z') \right) dF \\ &= \beta \int M(s,s',z,z') (1 + x[s,z]) \left(z' - x' + \hat{Q}' \right) dF \end{aligned} \tag{27}$$

where

$$M(s,s',z,z') \equiv \left(\frac{[1 + x(s,z)] (z' - x[s'(s,z), z'])}{z - x(s,z)} \right)^{-\sigma}. \tag{28}$$

is the MRS in consumption between today and tomorrow. This is the same as in Lucas (1978), but adjusted for the growth in the capital stock and, hence, in dividends.

Firm's choice of x.—Starting in the state (s,z), the firm must choose the same x that all other firms choose. Let us use bold letters to denote aggregate states and decisions. The firm is also concerned with the dividend it pays its current shareholders and therefore acts so as to maximize its cum-dividend value. That value is $\hat{Q}(s,z)k + zk - xk$. We put in bold letters the decision rules of other firms $\mathbf{x}(\mathbf{s},z)$ and $\mathbf{s}'(\mathbf{s},z)$ (they are the same decision, of course). We also must allow for the possibility that the firm in question can let its (k,s) evolve differently from that of other firms. Now, $\mathbf{k}$ affects $\hat{Q}$ only through its effect on $U'(C')/U'(C)$.

Let P denote the cum-dividend price of $1/k$'th of the representative firm, i.e., the price of the tuple $(1,s)$. Equilibrium is efficient if $P = v_k + s v_S$, with v defined in (6). The functional equation (in units of the consumption good) for its cum-dividend price per unit of k is

$$P(\mathbf{s},s,z) = \max_x \left(z - x + \beta (1 + x) \int M(\mathbf{s},\mathbf{s}',z,z') P(\mathbf{s}'[\mathbf{s},z], s', z) dF \right) \tag{29}$$

We are implicitly assuming that even if a firm were to choose a value of s different from what other firms choose, the market is fully aware of it, and prices the firm accordingly. In other words, although markets are incomplete, everyone knows each firm's (k,s,z). In equilibrium,

1. All firms must choose the same x, and so we ask that in state $(s,z) = (\mathbf{s},z)$, the firm will behave like other firms. That is, at the fixed point for P, the RHS of (29) is maximized by $x(s,s,z) = \mathbf{x}(s,z)$. This would imply that $s' = \frac{\lambda+s-x(s,s,z)}{1+x(s,s,z)} = \mathbf{s}'(s,z) = \frac{\lambda+s-\mathbf{x}(s,z)}{1+\mathbf{x}(s,z)}$.

2. For all (s,z), the maximized value of the firm must equal the value that the shareholders hold:

$$P(\mathbf{s},s,z) = z - x(\mathbf{s},s,z) + (1 + x(\mathbf{s},s,z))\,\hat{Q}(\mathbf{s},z)\,. \tag{30}$$

In fact, property 1 implies property 2 as one can deduce by setting $x(s,s,z) = \mathbf{x}(s,z)$ for all (s,z) so that $s' = \mathbf{s}'$, in which case substitution from (30) into (29) makes it identical to (27). Thus it suffices to show that property 1 holds. Recall that $U(C) = \frac{c^{1-\sigma}}{1-\sigma}$ so that $U'(C')/U'(C) = [(1+x)(z'-x')/(z-x)]^{-\sigma}$. Then, evaluated at $x = \mathbf{x}$, the FOC in (29), calculated by solving

$$P(\mathbf{s},s,z) = \max_{s'}\left\{z - \hat{x}(s',s) + \beta(1+\hat{x}[s',s])\int M(\mathbf{s},\mathbf{s}'[\mathbf{s},z],z,z')\,P(\mathbf{s}'[\mathbf{s},z],s',z')\,dF\right\} \tag{31}$$

where

$$\hat{x}(s',s) = \frac{\lambda+s-s'}{1+s'}, \tag{32}$$

and does not depend on the firm's action.

Differentiability of P.—Similar to the proof of Lemma 4 we can establish that $P_2(\mathbf{s},s,z) \equiv \frac{\partial}{\partial s}P(\mathbf{s},s,z)$ exists everywhere. Since

$$\frac{\partial\hat{x}}{\partial s'} = \frac{\partial(1+\hat{x})}{\partial s'} = \frac{\partial}{\partial s'}\left(\frac{1+\lambda+s}{1+s'}\right) = -\frac{1+x}{1+s'}$$

the derivative w.r.t. s' is $\frac{1+x}{1+s'} - \beta\frac{1+x}{1+s'}\int M'P'dF + (1+x)\beta\int M'P_2'dF \le 0$, with an exact equality if $s' > 0$. The term $(1+x)$ cancels, and so the FOC to the problem (31) is

$$1 - \beta\int MPdF + (1+s')\beta\int MP_2dF\begin{cases} = 0 & \text{if } s' > 0 \\ \le 0 & \text{if } s' = 0\end{cases}, \tag{33}$$

Efficiency.—Here P is the cum-dividend price of one-k'th of the firm in current consumption units. Per unit of its k, a firm is a package of $(1,s)$ units of (k,S). Therefore, efficiency would appear to require that $P = \frac{1}{U'}(v_k + sv_S)$. In what follows we let $x(s,z)$ denote the planner's optimal policy, and $s'(s,z) = \frac{\lambda+s-x(s,z)}{1+x(s,z)}$.

The next claim states that if the representative firm used the planner's policy, its market value would equal the marginal social value of the bundle (k,S):

Lemma 6

$$P(s,s,z) = P, \tag{34}$$

where P *is given in (22).*

Proof. Updating (34) by a period we have $P(s'[s,z], s', z') = (1-\sigma)(z' - x[s', z'])^{\sigma} w(s', z')$. Substituting into the RHS of (29), the latter becomes

$$\begin{aligned}
& z - x(s,z) + \beta(1 + x[s,z]) \int M(s, s', z, z')(1-\sigma)(z - x[s,z])^{\sigma} w(s'[s,z], z')\, dF \\
= \quad & z - x + (1-\sigma)\beta \frac{(1+x)^{1-\sigma}}{(z-x)^{-\sigma}} \int w(s', z')\, dF \quad \text{in view of (28)} \\
= \quad & (1-\sigma)(z - x[s,z])^{\sigma} w(s,z) \\
= \quad & P(s,s,z), \quad \text{as claimed in (34).}
\end{aligned}$$

■

The previous lemma is, however, conditional on the assumption that the representative firm uses the planner's policy, i.e., that

$$x(s,s,z) = x(s,z). \tag{35}$$

Next we shall show that (35) does hold if (34) does.

Lemma 7 *If P satisfies (34), then (35) holds.*

Proof. If (35), then the firm's FOC, (33), must coincide with the planner's FOC, (9). In view of (28), LHS of (33) can be written as $1 - \beta \int \left(\frac{(1+x)(z'-x')}{z-x} \right)^{-\sigma} (P' - [1+s'] P_2')\, dF$. This is the same as the LHS of (9) if

$$(z' - x')^{-\sigma} (P' - [1+s'] P_2') = \left[(z'-x')^{-\sigma} (1+z') + \lambda w_s' \right]$$

i.e., if

$$1 + z + \frac{\lambda w_s}{(z-x)^{-\sigma}} = P - (1+s) P_2 \tag{36}$$

Now applying the envelope theorem in (31) and noting that, since $\hat{x}(s', s) = \frac{\lambda + s - s'}{1+s'}$, $\frac{\partial \hat{x}}{\partial s} = \frac{1}{1+s'} = \frac{1+x}{1+\lambda+s}$, gives

$$\begin{aligned}
P_2 &= \frac{\partial \hat{x}}{\partial s}\left(-1 + \beta \int M' P' dF \right) = \frac{1+x}{1+\lambda+s}\left(-1 + \frac{P - (z-x)}{1+x} \right), \\
&= \frac{P - 1 - z}{1+\lambda+s}.
\end{aligned}$$

Substituting this into (36) for P_2 gives

$$\begin{aligned}
1 + z + \frac{\lambda w_s}{(z-x)^{-\sigma}} &= P - (1+s)\frac{P-1-z}{1+\lambda+s} \\
&= \frac{\lambda P}{1+\lambda+s} + (1+s)\frac{1+z}{1+\lambda+s}
\end{aligned}$$

Rearranging,

$$\frac{\lambda w_s}{(z-x)^{-\sigma}} = \frac{\lambda P - \lambda(1+z)}{1+\lambda+s},$$

i.e.,

$$\begin{aligned} w_s &= (z-x)^{-\sigma}\frac{P-(1+z)}{1+\lambda+s} \\ &= (z-x)^{-\sigma}\frac{(1-\sigma)(z-x[s,z])^{\sigma} w(s,z) - (1+z)}{1+\lambda+s} \qquad (34), \\ &= \frac{1}{1+\lambda+s}\left([1-\sigma]w - (1+z)[z-x]^{-\sigma}\right) \end{aligned}$$

But this is the same as (8). ■

Proposition 4 *Equilibrium is efficient.*

Proof. The RHS of (31) is a contraction operator (NEED PRIMITIVE CONDITIONS SO THAT $\beta(1+x)\int MdF < 1-\varepsilon$ for some $\varepsilon > 0$). By Lemma 8 its unique solution, $P(\mathbf{s}, s, z)$, satisfies (34) and by Lemma 9 it also satisfies (35), i.e., equilibrium decisions are efficient. ■

4.0.3 The effects of financial-market completion

The results say that if all firms are publicly traded, a stock market exists, the emergence of a seeds market should affect neither prices nor quantities. It is enough that all firms trade on the stock market. Even in a financially developed society like the U.S., however, only about one half of the privately owned capital does trade on stock markets; and therefore further enlargement of the stock market would probably raise efficiency, as Greenwood and Jovanovic (1990) argued. The model assumes that firms have identical z's, and identical s's, but firms can choose to differ from the rest, but that would lower their market value.

The efficiency result should extend to a situation in which firms do differ because, e.g., they draw different z's. Jovanovic and Braguinsky (2004) develop a related one-period model in which firms differ in two dimensions: Project quality which we can interpret as s, and managerial ability, which we can interpret as z. They find that even when s is private information to the firm being acquired, the stock market achieves efficiency.

All this must be qualified by noting that seeds, S , do not share some of the features of inventions that are sometimes thought important. Namely,

1. Seeds are of purely private value, and not costlessly reproducible – as information perhaps is – and cannot raise output in more than one firm;

2. The producer of a seed has a perfect property right to it even when markets for seeds do not exist

If either assumption did not hold we would expect efficiency to fail.

5 Numerical solution and fitting the data

Process for z.—We use the output-capital ratio for z. When de-trended linearly it follows an AR(1) process with autocorrelation coefficient 0.897, and innovation variance 0.026. The Tauchen-Hussey procedure for discretizing the AR yields a first-order Markov chain with 3 states, $z \in \{0.106, 0.152, 0.198\}$, and the symmetric transition probability matrix

z_1	z_2	z_3
0.788	0.210	0.004
	0.667	0.168
		0.788

(37)

Table 1: The Matrix of transition probabilities for z

which has the stationary distribution $(0.307, 0.386, 0.307)$.

Parameters.—At this point we assume that k depreciates at the rate δ and S at the rate γ so that their laws of motion (1) and (3) become $k' = (1-\delta)k + X$ and $Y' = \lambda k + (1-\gamma)Y - X$ respectively. The details are in Appendix 2. The parameter values

β	σ	δ	γ	λ	$\bar{z}$	ρ	std(z)
0.95	2.0	0.08	0.15	0.1080	0.1520	0.8970	0.0261

Table 2: Parameter values

were chosen, among other reasons, so as to match (i) A growth of per-capita income of 1.4%, and (ii) An average level of Tobin's Q of 1.22 and (iii) Some properties of the Q and x series since WW2 which will be shown in Figure 5. Section 6.2 shows that for a constant-z economy the growth rate is bounded by $\lambda - \delta$, and that if $x < \lambda - \delta$, seeds accumulate indefinitely and the seeds constraint becomes irrelevant. The depreciation of S is γ and it was (rightly or wrongly) chosen based on estimates of private obsolescence of knowledge by Griliches, Pakes, Schankerman and others. Section 5 will briefly deal with what happens when $\gamma = 0$.

5.1 Simulated decision rules and Q.

For the parameter values and transition probabilities stated in Tables 1 and 2, Figure 4 plots the equilibrium Q, the decision rules and the value function. In all the plots, the variable on the horizontal axis is s, the beginning-of-period stock of seeds. We may summarize the plots as follows:

1. Panel 1 of Figure 4 plots Tobin's Q. As s gets large, $p(s,z) \to 0$ for all z, and therefore $Q(s,z) \to 1$. The maximal Q of 1.75 occurs when $s = 0$ and $z = z_3$.

2. The second panel plots investment, which responds more to s when z is high. At z_3 investment is constrained at low values of s. In particular, $x(s, z_3) = \lambda + s$ when s is close to zero, so that the initial slope of the red curve in Panel 2 is unity. When $z \in \{z_1, z_2\}$, however, x is never constrained and s then has a much smaller effect on it.

3. In Panel 3 we see the long-run distribution of seeds. Twenty-two percent of the time $s = 0$, and seeds never reach 50 percent of k. Indeed, illustrated in Figure 5, the simulated s peaks at 0.32 in the late 80's.

4. Finally, the last panel plots w which is negative (because σ exceeds unity) and increasing in s. The increase with s is sharper for higher levels of z because seeds are more valuable when z is high.

After s reaches 0.3, it makes very little difference to any of the variables.

5.2 Fitting the data

The state variables of the model are k, S, and z, and the decision variable is x. In addition, we focused on the price of seeds, p, but the real motivation for it is the role that p plays in the price of the firm that, in its ex-dividend form, we label as Q. Thus we shall fit the following post-war series: (i) The output-capital ratio, which in the model is z, (ii) The seed-capital ratio, s, (iii) The investment-capital ratio, x, and (iv) Tobin's q as measured by $P - z$, with P given by (22). In all four Panels the blue lines represent the model and red lines represent the data.

The variables were constructed as follows:

1. The red line in Panel 1 of Figure 5 plots $z = Y/k$ where Y = private non-farm output and k = non-farm stock of capital. The model has just 3 values of z to fit this with – this quantity is exogenous in the model, being drawn from the probability distributions implied by the transition matrix in (37).

2. Panel 2 plots the series for s implied by the model as the blue line. Panel 2 also plots several possible proxies for s, each constructed via the formula

$$s' = \frac{n + (1-\gamma)s - x}{1 - \delta + x}, \tag{38}$$

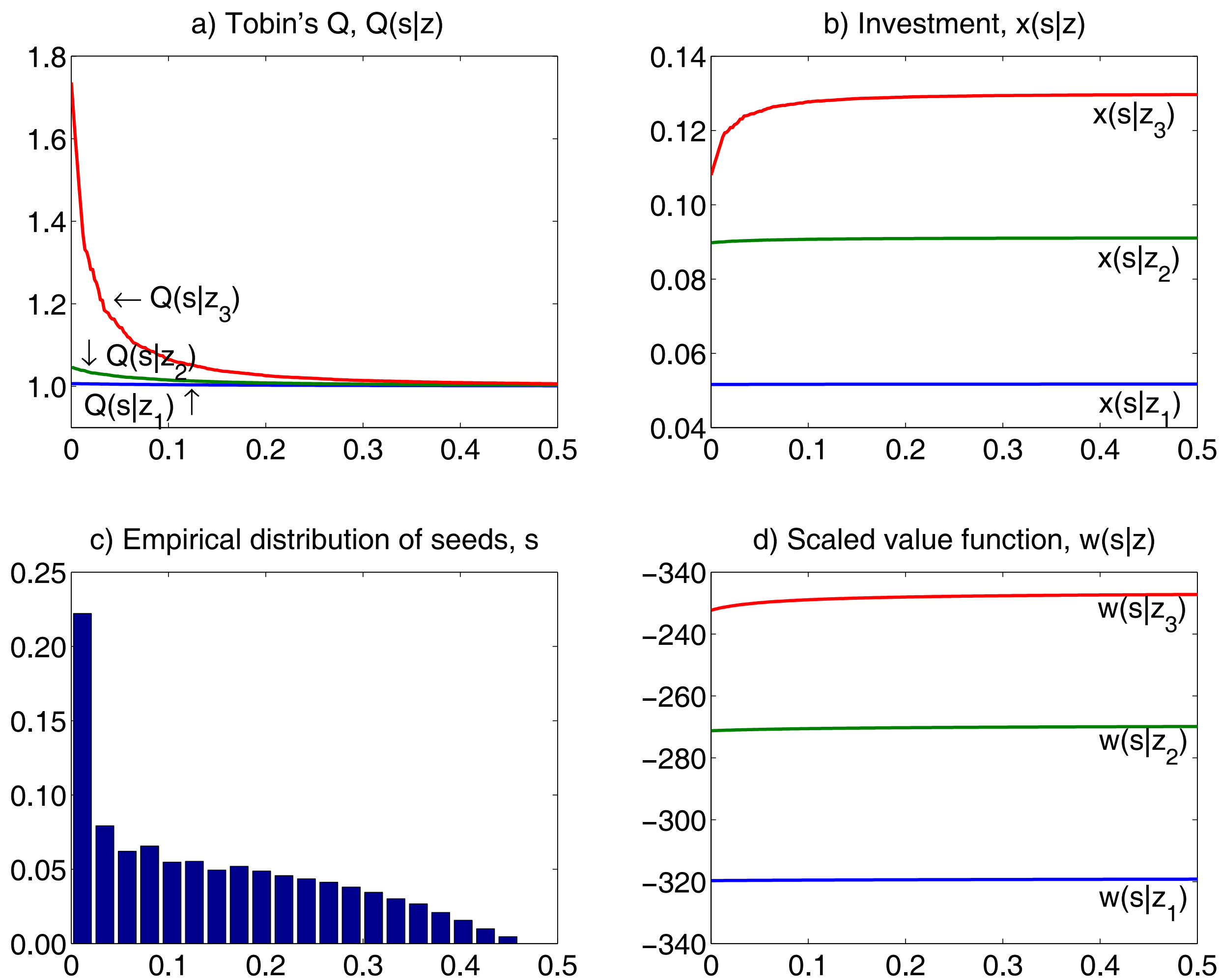

Figure 4: Simulated value, decision rules, and Tobin's Q for $z_1 < z_2 < z_3$. On the horizontal axis is s.

where n is one of the following: (A) n =patents/(θk) (red line), (B) n =trademarks/(θk) (green line), (C) $n = \lambda$ (turquoise line). The constant θ fixes units appropriately; it is explained in Appendix 3.[5] Red, green, and turquoise lines in the second panel of Figure 5 correspond to cases A, B, and C. In cases A and B the least-squares routine chose $s_0 = 0$ as the initial condition. we noted that the simulated s peaks at 0.32 in the late 80's and grossly overpredicts the estimate of s_t that produces the model's best fit to the post-war data – none of the estimates of s_t ever rises above 0.2, i.e., above twenty percent of installed capital.

3. Panel 3 shows that in its desire to fit Q and the output-capital ratio, the model generates too volatile an investment series, and z exerts a more important influence on it than does s. From Panel 2 we see that the simulated s peaks at 0.32

4. In Panel 4 of Figure 5 we plot the actual and fitted Q. For the measured Q, for 1951-1999 we use Hall's series, but since it ends in 1999, for the period 1999-2004 we use Abel's data scaled so that the two Q series match in 1999. This is the red line in Panel 4 of Figure 5. To get a sustained rise in Q we must have a prolonged period during which $z = z_3$. The '90s appear to have been such a period. The simulation has $z = z_3$ in periods 18-20, and Panel 4 shows that Q rises sharply during these periods as seeds get depleted, but Q then falls even more sharply when z reverts to its mean value of z_2 in period 21.

The parameters θ and s_0 were chosen to minimize the RSS between the simulated and constructed series. The model has a problem with reconciling the following facts:

- Y/k falls dramatically in the late 70's and early 80's, something that the model interprets as a low-z epoch causing the huge buildup of seeds portrayed in panel 2 and the resulting collapse of Q to its lowest possible level of unity, and
- The rise in Q starting in the early 80s. Even with the accompanying rise in the estimate of z from z_1 to z_2 in the mod 80s and then to z_3 in 1991, it takes time for s to be drawn to zero and for Q to rise to its maximal value of 1.75.

[5] The model is neutral in (λ, θ). Doubling these two parameters and doubling S_0 doubles S_t for all t but leaves all the other variables unchanged. Therefore θ has been normalized to unity up to this point.

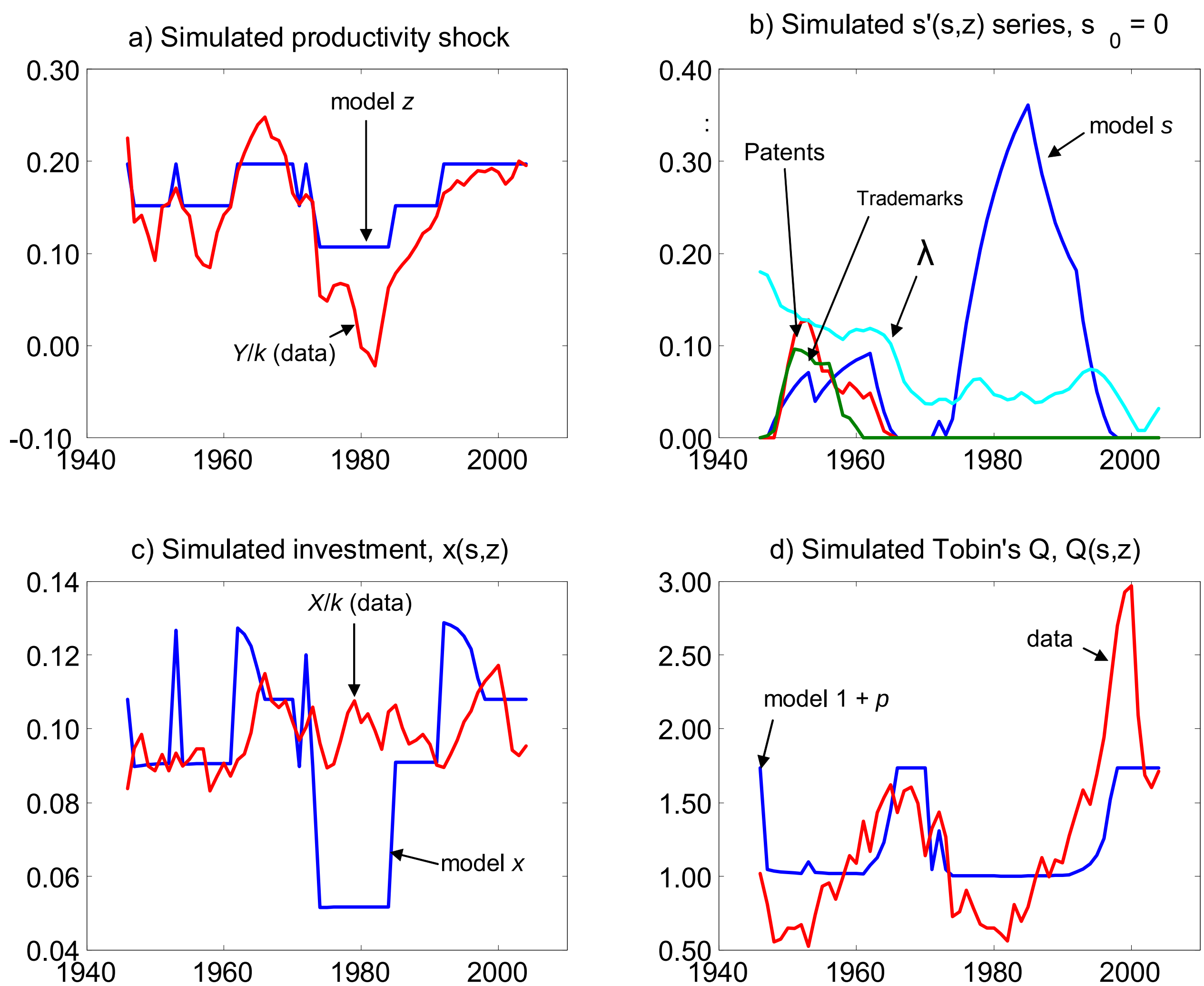

Figure 5: Fitting the model to post-war data

5.3 Intangibles and Q, and other unconditional correlations

The matrix of unconditional correlations for the model and data is shown in Table 3:

z	s	x	Q
	-0.62	0.96	0.69
		-0.54	**-0.57**
			0.46

Model

z	s	x	Q
	0.02	0.19	0.63
		-0.47	**-0.35**
			0.53

Data

Table 3: The Matrix of unconditional correlations in the model and in the data

The signs the model produces are mostly correct, but the magnitudes are far apart in some cases.

Intangibles and Q.—The correlation between s and Q is bolded in the two tables. A rise in s represents a rise in the ratio of unimplemented intangible capital to tangible capital. The stock of all intangible capital is $k + S$ with k being the number of seeds already in the ground and being used for production. Therefore the ratio

$$\frac{\text{All intangible capital}}{\text{Tangible capital}} = \frac{k+S}{k} = 1 + s$$

is also monotone in s. Herein lies the reason for why this model implies a fall in Q whereas Hall's (2000) implies a rise in Q. In my model, variation in intangibles is caused by variation in the stock of unimplemented seeds. In Hall's model there are variable proportions between intangibles and physical capital in production and there is no storage of intangibles, hence a rise in intangibles gives a rise in the productivity of the firm's measured capital and (barring GE effects) a rise in the firm's Q.

The model matches well the strong positive correlation between z and Q and the negative correlation between s and x. That the latter should be negative in the model may at first seem to contradict Corollary 2 which says that the policy $x(s, z)$ is increasing in s, a fact that is also borne out by Panel 2 of Figure 4. It turns out that the negative feedback effect of x on s' via () is stronger and renders the correlation negative.

We already saw that the model generates too much volatility in investment. To this it is driven by the attempt to also fit Q. But the signs of the $z - x$ correlations are both positive in the two tables, even if their magnitudes differ a lot. The glaring discrepancy is the negative relation between s and z. Like the negative relation between s and x, this one arises because s is constructed using (38), and again reflects the negative effect that x exerts on s' through this accounting relation, and not any negative effect of s on x.

5.4 Simulations and data fitting when $\gamma = 0$

The depreciation of S is γ and it was set at 0.15. But one can make the case that γ should be zero, since seeds represent knowledge and since in the model seeds are a private good and cannot be used by anyone else. In any case, the model generates more action for seeds in this case. When seeds last for ever, they exert a stronger influence on the decision rules and equilibrium prices. For the parameter values and transition probabilities stated in Tables 1 and 2 except that $\gamma = 0$, Figure 6 plots the equilibrium Q, the decision rules and the value function. In all the plots, the variable on the horizontal axis is s, the beginning-of-period seeds.

A comparison of Figures 4 and 6 shows, in their first Panels, Q responding more strongly to s; the maximal Q now is 2.6. Investment responds more strongly to s too, as does w. In Panel 3 we see the long-run distribution of seeds moving more to the right, so that $s = 0$ only seventeen percent of the time now as opposed to twenty-two percent previously.

A comparison of Figures 5 and 7 shows, in Panel 1, a slightly higher z now. The x series in Panel 3 is slightly less volatile. The main change is that now the model gets more volatility in Q, but still cannot really fit well the rise in Q starting in the early 80s.

6 Comparison with the standard growth model

It may be of interest to compare things to what the standard model says. The standard model obtains when $z_{\max} < \lambda$ or when s_0 is "large". I start with a comparison using the simulations, and then move to the analytics which are relatively easy to compare in the deterministic case

6.1 Simulations

If we simulate the model for a finite number of periods – 1000 in this case – there exists a s_0 so large that (4) will not bind, and so that the standard model roughly obtains over the 10,000 periods. We therefore let $s_0 = 50,000$ and compare that with the model when $s_0 = 0$. We want, however, that the growth rate should be the same in the two cases. When s_0 is large and when (4) does not not bind, the rate of growth is $E(x) - \delta$. When s_0 is low, and when (4) occasionally binds, investment will on average be lower and so will growth, unless we change some parameters so as to compensate. We lower z therefore by a constant when s_0 is large so as to keep average growth at 1.38 percent. Thus, for two separate initial conditions, $s_0 = 0$ and $s_0 = 50000$, we take a 1000-long random sequence of the z's and calculate the two sets of moments in Table 4:

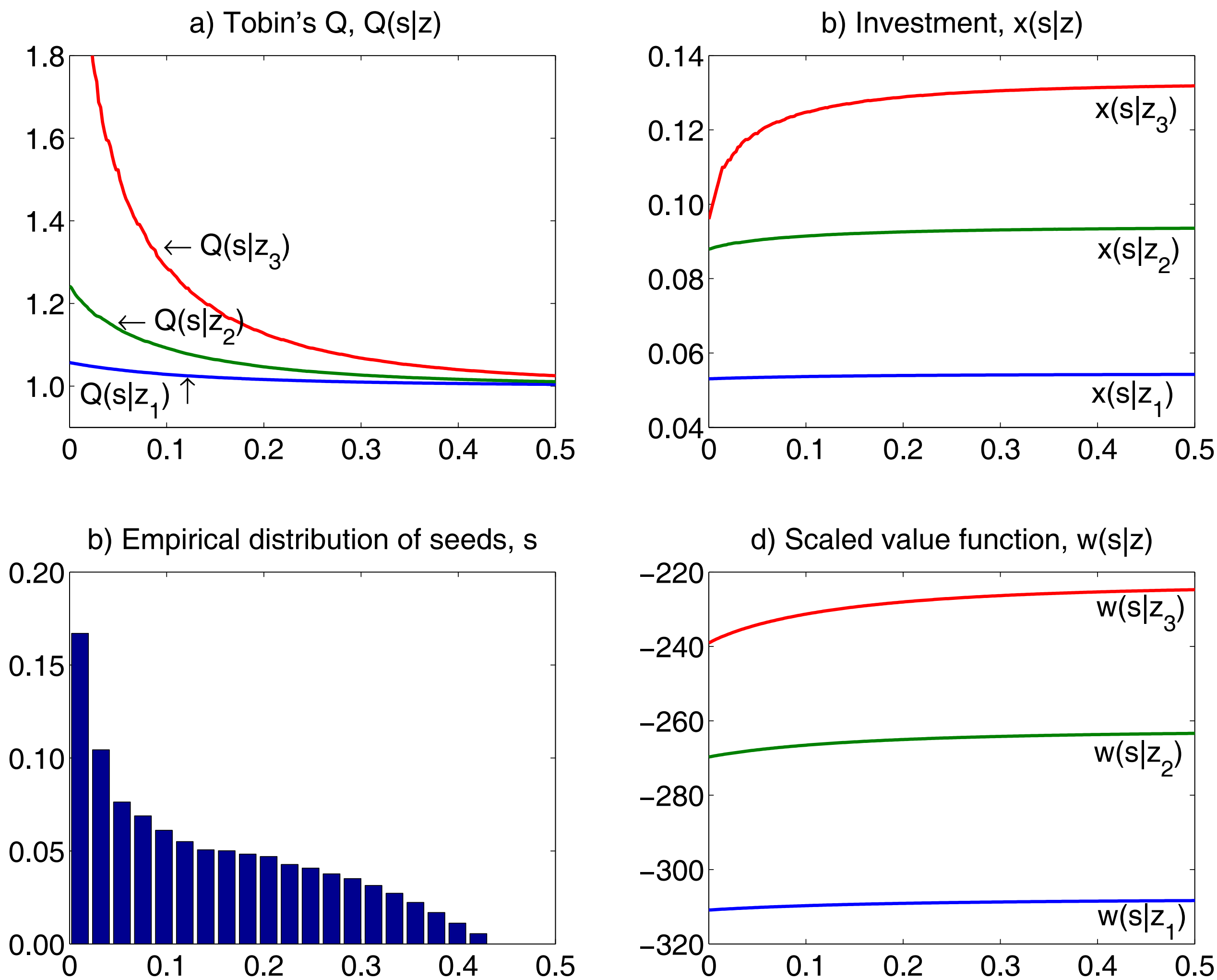

Figure 6: Decision rules and Q when $\gamma = 0$.

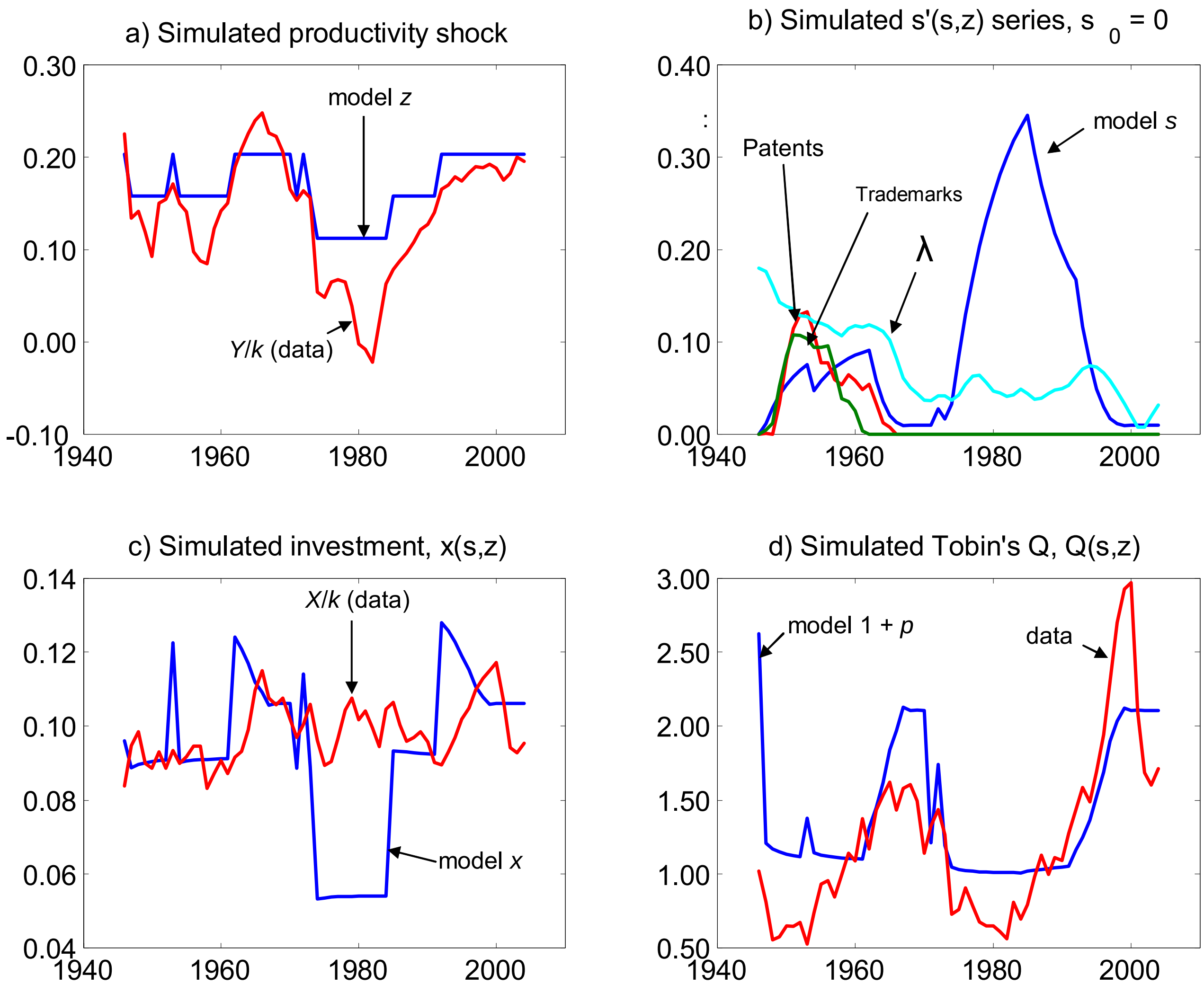

Figure 7: Fitting to data when $\gamma = 0$

Stat	$s_0 = 0$	$s_0 = 50,000$
$E(z)$	0.1629	0.1484
$E(x)$	0.0938	0.0937
$E(s)$	0.0978	5235.4
$E(Q)$	1.2181	1.0002
$E(c)$	0.0690	0.0546
$\sigma(z)$	0.0343	0.0343
$\sigma(x)$	0.0241	0.0301
$\sigma(s)$	0.1117	10725.3
$\sigma(Q)$	0.3073	0.0000
$\sigma(c)$	0.0129	0.0043

.*Table 1*: COMPARISON TO THE STANDARD GROWTH MODEL

On the region where (4) binds, consumption is more volatile than in the standard model. The volatility of z is the same in the two cases. The case $s_0 = 0$ is portrayed in Figure 5. When we raise s_0 to $50,000$, the standard deviation of c, $\sigma(c)$, falls by a factor of 3.5, while the standard deviation of x rises by a factor of just 1.4. In other words, the $s_0 = 0$ version retains most of the volatility of investment implied by a model with *no* adjustment costs. Note, moreover, that the removal of the constraint comes at the cost of underpredicting Q. It would seem, then, that the current setup is a desirable alternative to convex-adjustment-cost models that also fit Q. In order to fit average Q, they reduce the volatility of investment.

6.1.1 Deterministic case

Suppose z is a constant. Let

$$x = \frac{X}{k} \quad \text{and} \quad s = \frac{S}{k}.$$

Since k does not depreciate, x then equals the growth rate of k and of C. Let's solve for the constant-growth rate that would obtain in the absence of the constraint (4). We shall call this the "desired" growth rate, x^d. Then $U'(C_{t+1})/U'(C_t) = (1+x)^{-\sigma}$ and the effective discount factor is

$$\hat{\beta} \equiv \beta (1+x)^{-\sigma}. \tag{39}$$

An additional unit of capital produces z units for ever, and so optimal investment leads to a Tobin's Q of unity:

$$Q \equiv \left(\frac{\hat{\beta}}{1-\hat{\beta}}\right) z = 1. \tag{40}$$

Equations (39) and (40) can be solved for x^d:

$$1 + x^d = (\beta [1+z])^{1/\sigma}. \tag{41}$$

The model collapses to the standard model if s goes off to infinity. We seek parameter restrictions that will prevent this from happening. From (4),

$$x_t \leq \min(z, \lambda + s_t) \tag{42}$$

This, however, is a short-run constraint, that holds at each t. If k were to grow faster than λ, s_t would eventually become negative. To see this, combine (3) and (2) to get $S' = S - X + \lambda k$ and, hence,

$$s_{t+1} = \frac{\lambda + s_t - x_t}{1 + x_t}. \tag{43}$$

It's easy to show that λ is the maximal feasible long-run growth rate. Let ε be a constant, and suppose that $x = \lambda + \varepsilon$. Then

Lemma 8 *For all $s_0 \geq 0$,*

$$\begin{array}{lrcl} (i) & \varepsilon & > & 0 \Longrightarrow s_t \to -\infty \\ (ii) & \varepsilon & < & 0 \Longrightarrow s_t \to +\infty \end{array}$$

Proof. (i) Let $\varepsilon > 0$. Then $s_{t+1} = \frac{\lambda+s_t-x}{1+x} = \frac{s_t-\varepsilon}{1+x} < s_t - \frac{\varepsilon}{1+x}$, so that $s_t < s_0 - \left(\frac{\varepsilon}{1+x}\right) t \to -\infty$. (ii) let $\varepsilon < 0$. Then $s_{t+1} > s_t + \frac{|\varepsilon|}{1+x}$ so that $s_t > s_0 + \frac{|\varepsilon|}{1+x} t \to +\infty$. ■

Desired growth exceeds λ if

$$[\beta(1+z)]^{1/\sigma} > 1 + \lambda,$$

which is also when the seeds constraint binds in every period. High values of z or β, and low values of σ and λ make it more likely that this inequality will hold. Tobin's Q is just the present value of the marginal products of capital, $\Sigma_{t=1}^{\infty} \tilde{\beta}^t z$, i.e.,

$$Q = \left(\frac{\tilde{\beta}}{1-\tilde{\beta}}\right) z, \qquad \text{where } \tilde{\beta} = \beta(1+\lambda)^{-\sigma} > \beta\left(1+x^d\right)^{-\sigma} = \hat{\beta}.$$

Values of Q above unity arise because consumption growth is lower than it would be under x^d; the rate of interest is thus lower, and this raises the present value of income from capital above its cost.

The case $\sigma = 1$.—From (41), the desired investment and growth rate x is

$$x^d(z) = \beta z - (1-\beta),$$

and Tobin's Q is

$$Q(z) = \begin{cases} 1 & \text{if } x^d(z) \leq \lambda \\ \frac{\beta}{1+\lambda-\beta} z & \text{if } x^d(z) > \lambda \end{cases}.$$

The value of z at which $x^d(z) = \lambda$ is $\frac{1}{\beta}(1+\lambda-\beta)$. Figure 8 plots $x^d(z)$ and $Q(z)$. Of course, $x = \min\left(\lambda, x^d[z]\right)$.

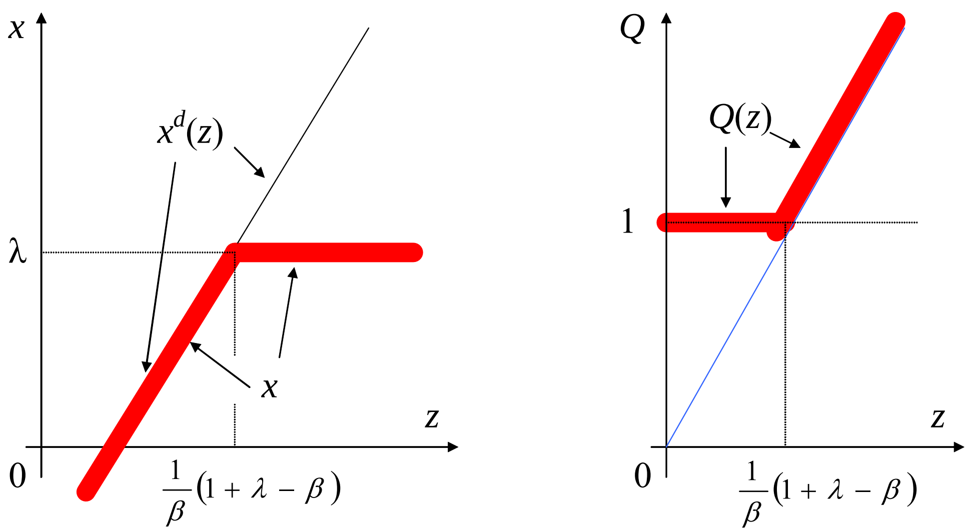

Figure 8: COMPARATIVE STEADY STATES FOR x AND q WHEN $\sigma = 1$.

Transitional dynamics in the deterministic case These are easier to analyze if time is continuous. Let (k_0, S_0) be given, with $S_0 > 0$. Let preferences be

$$\int_0^\infty \frac{1}{1-\sigma} e^{-\rho t} C_t^{1-\sigma} dt.$$

Output is

$$zk = C + X,$$

the seed constraint reads

$$X \leq S$$

and the laws of motion are

$$\dot{S} = \lambda k - X$$

and

$$\dot{k} = k + X$$

We know that in the absence of the seed constraint,

$$\frac{\dot{C}}{C} = \frac{\dot{k}}{k} = \frac{z-\rho}{\sigma}$$

and so we shall assume that

$$\lambda < \frac{z-\rho}{\sigma} \tag{44}$$

so that eventually the seed constraint must bind, and so that eventually we know that $X = \lambda k$. But we want to see how fast this happens from initial conditions. We especially want the time path of Tobin's Q, defined here as the discounted marginal product of k:

$$Q = z\int_t^\infty e^{-\rho(\tau-t)}\frac{U'(C_\tau)}{U'(C_t)}d\tau$$

In the limit, consumption will grow at the rate λ so that $\frac{U'(C_\tau)}{U'(C_t)} = e^{-\sigma\lambda(\tau-t)}$ and Q will converge to

$$Q_\infty = z\int_t^\infty e^{-(\rho+\sigma\lambda)(\tau-t)}d\tau = \frac{z}{\rho+\sigma\lambda}$$

where the rate of interest is

$$\rho + \sigma\lambda$$

which is less than z if (44) holds, so that $Q_\infty > 1$. But if (44) does not hold, then consumption grows at the rate $\frac{z-\rho}{\sigma}$ and $Q_\infty = 1$.

Well, this is just a version of the exhaustible-resources problem. The Hamiltonian is

$$\frac{(zk-X)^{1-\sigma}}{1-\sigma} + \mu X + m(\lambda k - X) + n(S-X)$$

(I think we can just add $n(S-X)$ as I did at the end). The optimality conditions are

$$\begin{array}{rcr} X & : & -(zk-X)^{-\sigma} + \mu - m - n = 0 \\ k & : & z(zk-X)^{-\sigma} + \lambda m = -\dot{\mu} + \rho\mu \\ S & : & n = -\dot{m} + \rho m \end{array}$$

and the two constraints must hold.

The region $[0,T)$ *where* $X < S$.—This is the initial stage, for a finite time, call it $[0,T]$. In this region, $n = 0$ so that the last condition implies

$$m_t = m_0 e^{\rho t} \qquad \text{for } t < T$$

The first condition implies, on this region,

$$(zk-X)^{-\sigma} = \mu - m.$$

Substituting all this into the middle condition gives us

$$z\left(\mu - m_0 e^{gt}\right) + \lambda m_0 e^{\rho t} = -\dot{\mu} + \rho\mu$$

which is the differential equation

$$\dot{\mu} = (\rho - z)\mu + (z-\lambda)m_0 e^{\rho t}$$

Now an equation of the form $\frac{dx}{dt} = Ax + Be^{\rho t}$ has the solution $x = C_1 e^{At} + B\frac{e^{\rho t}}{\rho - A}$. Therefore

$$\mu_t = C_1 e^{(\rho-z)t} + \frac{(z-\lambda)m_0}{1+z}e^{\rho t}$$

The region $[T, \infty)$.—Here all the multipliers are constant. In particular

$$\mu = Q_\infty = 1 + m$$

These two conditions should help us determine T, etc..

7 Conclusion

This paper has emphasized extensive-margin investment. We found that when the TFP shock is good, investment is likely to be constrained from above, so that consumption rises more than it otherwise would. We also found that having a lot of unimplemented intangibles on hand lowers stock prices, just as a favorable "investment-specific technological shock," does, but that the model also has an intertemporal substitution components that is missing in other models that have such shocks. Finally, we found that a stock market alone suffices to ensure efficiency of the equilibrium.

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8 Appendix

8.0.2 Proof of differentiability (Lemma 4)

I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

$$s'_s = \frac{\lambda + s - x_s}{1 + x_s} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}.$$

Variations.—We use (32) to figure out the feasible variations.

Variation (i).—If we begin at state $s+h$, and if we want to end up at s'_s, we need an investment of

$$\begin{aligned}
\hat{x}\left(s'_s, s+h\right) &= \frac{\lambda + s + h - \frac{\lambda+s-x_s}{1+x_s}}{1 + \frac{\lambda+s-x_s}{1+x_s}} = \frac{\left(1+x_s\right)\left(\lambda+s+h\right) - \left(\lambda+s-x_s\right)}{1 + x_s + \lambda + s - x_s} \\
&= \frac{\left(1+x_s\right)h + x_s\left(\lambda+s\right) + x_s}{1+\lambda+s} \\
&= x_s + h\frac{1+x_s}{1+\lambda+s}.
\end{aligned}$$

Then

$$A_h \equiv \left(\frac{1+\hat{x}}{1+x_s}\right)^{1-\sigma} = \left(1 + \frac{h}{1+\lambda+s}\right)^{1-\sigma},$$

and

$$\hat{x} - x_s = h\frac{1+x_s}{1+\lambda+s}.$$

Therefore

$$\begin{aligned}
w\left(s+h, z\right) &\geq U\left(z - \hat{x}\left[s'_s, s+h\right]\right) + \left(1 + \hat{x}\left[s'_s, s+h\right]\right)^{1-\sigma}\beta\int w\left(s'_s, z'\right)dF \\
&= U\left(z - \hat{x}\left[s'_s, s+h\right]\right) + A_h\left(1+x_s\right)^{1-\sigma}\beta\int w\left(s'_s, z'\right)dF \\
&= U\left(z - \hat{x}\left[s'_s, s+h\right]\right) + A_h\left(w\left(s,z\right) - U\left(z - x_s\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
w(s+h,z) - w(s,z) &\geqslant U(z - \hat{x}[s'_s, s+h]) - A_h U(c_s) + (A_h - 1)w(s,z) \\
&= U(z - \hat{x}[s'_s, s+h]) - U(c_s) + (A_h - 1)(w(s,z) - U(c_s)).
\end{aligned}$$

Dividing both sides by h and taking the limit as $h \searrow 0$ gives

$$\begin{aligned}
\frac{d}{ds}w(s,z) &\geqslant -U'(c_s)\lim_{h\searrow 0}\frac{\hat{x} - x_s}{h} + \lim_{h\searrow 0}\frac{(A_h - 1)}{h}\left[w(s,z) - U(c_s)\right] \\
&= -U'(c_s)\frac{1+x_s}{1+\lambda+s} + (1-\sigma)\frac{w(s,z) - U(c_s)}{1+\lambda+s}. \qquad (45)
\end{aligned}$$

because, by L'Hôpital's rule,

$$\begin{aligned}\lim_{h\searrow 0}\frac{(A_h-1)}{h} &= \lim_{h\searrow 0}\frac{dA_h}{dh}=\lim_{h\searrow 0}\frac{d}{dh}\left(1+\frac{h}{1+\lambda+s}\right)^{1-\sigma}\\ &= \frac{1-\sigma}{1+\lambda+s}\lim_{h\searrow 0}\left(1+\frac{h}{1+\lambda+s}\right)^{-\sigma}\\ &= \frac{1-\sigma}{1+\lambda+s}\end{aligned}$$

Variation 2: Start from s and end at s'_{s+h}...

Variation (ii) .— If we begin at state s, and if we want to end up at s'_{s+h}, we need an investment of

$$\begin{aligned}\hat{x}\left(s'_{s+h},s\right) &= \frac{\lambda+s-\frac{\lambda+s+h-x_{s+h}}{1+x_{s+h}}}{1+\frac{\lambda+s+h-x_{s+h}}{1+x_{s+h}}}=\frac{(1+x_{s+h})(\lambda+s)-(\lambda+s+h-x_{s+h})}{1+x_{s+h}+\lambda+s+h-x_{s+h}}\\ &= \frac{x_{s+h}(\lambda+s)-(h-x_{s+h})}{1+\lambda+s+h}=\frac{(1+\lambda+s)x_{s+h}-h}{1+\lambda+s+h}\\ &= \frac{(1+\lambda+s+h)x_{s+h}-h(1+x_{s+h})}{1+\lambda+s+h}\\ &= x_{s+h}-\frac{h(1+x_{s+h})}{1+\lambda+s+h} \qquad (46)\\ &< x_{s+h}-\frac{h(1+x_s)}{1+\lambda+s+h} \qquad (47)\end{aligned}$$

because by Corollary 2, x is increasing in s. We shall also need the following implication of (46):

$$B_h\equiv\left(\frac{1+\hat{x}}{1+x_{s+h}}\right)^{1-\sigma}=\left(1-\frac{h}{1+\lambda+s+h}\right)^{1-\sigma}$$

Therefore

$$\begin{aligned}w(s,z) &\geq U(z-\hat{x})+(1+\hat{x})^{1-\sigma}\beta\int w\left(s'_{s+h},z'\right)dF\\ &= U(z-\hat{x})+B_h(1+x_{s+h})^{1-\sigma}\beta\int w\left(s'_{s+h},z'\right)dF\\ &= U(z-\hat{x})-B_hU(z-x_{s+h})+B_hw(s+h,z).\end{aligned}$$

and therefore

$$w(s,z)-w(s+h,z)\geq U(z-\hat{x})-B_hU(z-x_{s+h})+(B_h-1)w(s+h,z),$$

i.e.,

$$\begin{aligned} w(s+h,z) - w(s,z) &\leq B_h U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h) w(s+h,z) \quad (48) \\ &= U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h)[w(s+h,z) - U(c_{s+h})] \quad (49) \end{aligned}$$

Now, $[w(s+h,z) - U(z - x_{s+h})]$ is Lipschitz in h for every $z > 0$. This is because it is bounded above by the increment in value when a unit of consumption is added in perpetuity, and the latter is bounded as long as $c > 0$, i.e., as long as $z > 0$). Now, by (47), $x_{s+h} \geq \hat{x} + \frac{h(1+x_s)}{1+\lambda+s+h}$ and therefore

$$U(z - x_{s+h}) - U(z - \hat{x}) \leq U\left(z - \hat{x} + \frac{h(1+x_s)}{1+\lambda+s+h}\right) - U(z - \hat{x})$$

Using the RHS of this expression to replace the first two terms on the RHS of 49) leaves the inequality in (49) undisturbed. Moreover, using L'Hôpital's rule as before,

$$\lim_{h \searrow 0} \frac{1}{h}(1 - B_h)[w(s+h,z) - U(c_{s+h})] = \frac{1-\sigma}{1+\lambda+s}[w(s,z) - U(c_s)]$$

Putting this all together,

$$w_s \leq \frac{1}{1+\lambda+s}\left(U'(c_s)(1+x_s) + (1-\sigma)[w(s,z) - U(c_s)]\right) \tag{50}$$

Then (57) and (50) imply (8). To see this, (8) says (in this notation) that

$$w_s = \frac{1}{1+\lambda+s}\left([1-\sigma]w - (1+z)U'(c)\right) > 0.$$

For them to be the same we would need that

$$-(1+x)U' + (1-\sigma)(w - U) = (1-\sigma)w - (1+z)U',$$

i.e.,

$$-(1+x)U' - (1-\sigma)U = -(1+z)U',$$

i.e.

$$(1-\sigma)U = (z-x)U'$$

which is true because $z - x = c$ so that both sides of the equation equal $c^{1-\sigma}$. Therefore (57) and (50) imply (8).

8.0.3 Depreciation

Let $\delta =$ depreciation of k and let γ be the depreciation of S. The laws of motion and the value are

$$k' = k(1-\delta) + X, \tag{51}$$

$$S' = S(1-\gamma) + \lambda k - X, \tag{52}$$

and

$$v(k,S,z) = \max_{X \leq \lambda k + S} \left\{ \frac{(zk - X)^{1-\sigma}}{1-\sigma} + \beta \int v\left(k[1-\delta] + X, \lambda k + S[1-\gamma] - X, z'\right) dF \right\}. \tag{53}$$

Since

$$\frac{S'}{k'} = \frac{S'}{k}\frac{k}{k'} = \frac{s(1-\gamma) + \lambda - x}{1-\delta+x},$$

we have

$$s' = \frac{\lambda + s(1-\gamma) - x}{1-\delta+x}, \tag{54}$$

so that $(1-\delta+x)s' = \lambda + s(1-\gamma) - x$. Collecting terms, we get

$$xs' + x = \lambda + s(1-\gamma) - (1-\delta)s',$$

which leaves us with

$$\hat{x}(s',s) = \frac{\lambda + s(1-\gamma) - (1-\delta)s'}{1+s'}. \tag{55}$$

The auxiliary Bellman equation is

$$w(s,z) = \max_x \left\{ \frac{(z-x)^{1-\sigma}}{1-\sigma} + (1-\delta+x)^{1-\sigma}\beta \int w\left(\frac{\lambda + s(1-\gamma) - x}{1-\delta+x}, z'\right) dF \right\}, \tag{56}$$

and we still have $P = \frac{v_k + s v_s}{C^{-\sigma}}$.

differentiability, i.e., w_s, when there is depreciation I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

$$s'_s = \frac{\lambda + s(1-\gamma) - x_s}{1-\delta+x_s} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + (s+h)(1-\gamma) - x_{s+h}}{1-\delta+x_{s+h}}.$$

If we begin at state $s+h$, and to end up at s'_s we need an investment of

$$
\begin{aligned}
\hat{x}\left(s'_s, s+h\right) &= \frac{\lambda + s\left(1-\gamma\right) - \left(1-\delta\right) s'_s}{1+s'_s} \\
&= \frac{\lambda + \left(s+h\right)\left(1-\gamma\right) - \left(1-\delta\right) \frac{\lambda+s(1-\gamma)-x_s}{1-\delta+x_s}}{1+\frac{\lambda+s(1-\gamma)-x_s}{1-\delta+x_s}} \text{ (substituting from [54])} \\
&= \frac{\left(1-\delta+x_s\right)\left(\lambda+\left(s+h\right)\left(1-\gamma\right)\right) - \left(1-\delta\right)\left(\lambda+s\left(1-\gamma\right)-x_s\right)}{1-\delta+x_s+\lambda+s\left(1-\gamma\right)-x_s} \\
&= \frac{\left(1-\delta+x_s\right)\left(\lambda+s\left(1-\gamma\right)+h\left(1-\gamma\right)\right) - \left(1-\delta\right)\left(\lambda+s\left(1-\gamma\right)-x_s\right)}{1-\delta+\lambda+s\left(1-\gamma\right)} \\
&= \frac{\left(1-\delta\right)\left(\lambda+s\left(1-\gamma\right)+h\left(1-\gamma\right)\right) + x_s\left(\lambda+s\left(1-\gamma\right)+h\left(1-\gamma\right)\right) - \left(1-\delta\right)\left(\lambda+s\left(1-\right.\right.}{1-\delta+\lambda+s\left(1-\gamma\right)} \\
&= \frac{\left(1-\delta\right)h\left(1-\gamma\right) + x_s\left(\lambda+s\left(1-\gamma\right)+h\left(1-\gamma\right)\right) + \left(1-\delta\right)x_s}{1-\delta+\lambda+s\left(1-\gamma\right)} \\
&= \frac{\left(1-\delta\right)h\left(1-\gamma\right) + x_s\left[1-\delta+\lambda+s\left(1-\gamma\right)+h\left(1-\gamma\right)\right]}{1-\delta+\lambda+s\left(1-\gamma\right)} \\
&= x_s + \frac{\left(1-\delta\right)h\left(1-\gamma\right) + x_s h\left(1-\gamma\right)}{1-\delta+\lambda+s\left(1-\gamma\right)} \\
&= x_s + h\frac{\left(1-\gamma\right)\left(1-\delta+x_s\right)}{1-\delta+\lambda+s\left(1-\gamma\right)}.
\end{aligned}
$$

Then

$$
A_h \equiv \left(\frac{1-\delta+\hat{x}}{1-\delta+x_s}\right)^{1-\sigma} = \left(1+\frac{h\frac{(1-\gamma)(1-\delta+x_s)}{1-\delta+\lambda+s(1-\gamma)}}{1-\delta+x_s}\right)^{1-\sigma} = \left(1+h\frac{1-\gamma}{1-\delta+\lambda+s\left(1-\gamma\right)}\right)^{1-\sigma},
$$

and

$$
\hat{x} - x_s = h\frac{\left(1-\gamma\right)\left(1-\delta+x_s\right)}{1-\delta+\lambda+s\left(1-\gamma\right)}.
$$

Therefore

$$
\begin{aligned}
w\left(s+h, z\right) &\geq U\left(z-\hat{x}\left[s'_s, s+h\right]\right) + \left(1-\delta+\hat{x}\left[s'_s, s+h\right]\right)^{1-\sigma}\beta\int w\left(s'_s, z'\right)dF \\
&= U\left(z-\hat{x}\left[s'_s, s+h\right]\right) + A_h\left(1-\delta+x_s\right)^{1-\sigma}\beta\int w\left(s'_s, z'\right)dF \\
&= U\left(z-\hat{x}\left[s'_s, s+h\right]\right) + A_h\left(w\left(s,z\right) - U\left(z-x_s\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
w(s+h,z) - w(s,z) &\geqslant U(z-\hat{x}[s'_s, s+h]) - A_h U(c_s) + (A_h - 1)w(s,z) \\
&= U(z-\hat{x}[s'_s, s+h]) - U(c_s) + (A_h - 1)(w(s,z) - U(c_s)).
\end{aligned}
$$

Dividing both sides by h and taking the limit as $h \searrow 0$ gives

$$\frac{d}{ds}w(s,z) \geqslant -U'(c_s)\lim_{h\searrow 0}\frac{\hat{x}-x_s}{h} + \lim_{h\searrow 0}\frac{(A_h-1)}{h}\left[w(s,z)-U(c_s)\right]$$
$$= -U'(c_s)\frac{(1-\gamma)(1-\delta+x_s)}{1-\delta+\lambda+s(1-\gamma)} + \frac{(1-\sigma)(1-\gamma)}{1-\delta+\lambda+s(1-\gamma)}\left[w(s,z)-U(c_s)\right] \tag{57}$$

because, by L'Hôpital's rule,

$$\begin{aligned}
\lim_{h\searrow 0}\frac{(A_h-1)}{h} &= \lim_{h\searrow 0}\frac{dA_h}{dh} = \lim_{h\searrow 0}\frac{d}{dh}\left(1+h\frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}\right)^{1-\sigma} \\
&= \frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}\lim_{h\searrow 0}\left(1+h\frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}\right)^{-\sigma} \\
&= \frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}.
\end{aligned}$$

Then

$$\begin{aligned}
w_s &= \frac{(1-\gamma)(1-\delta+x)}{1-\delta+\lambda+s(1-\gamma)}\left([1-\sigma]w-(1-\delta+x_s)U'-(1-\sigma)U\right) \\
&= \frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}\left([1-\sigma]w-(1-\delta+x)(z-x)^{-\sigma}-(z-x)^{1-\sigma}\right) \\
&= \frac{1-\gamma}{1-\delta+\lambda+s(1-\gamma)}\left([1-\sigma]w-(z-x)^{-\sigma}[1-\delta+z]\right),
\end{aligned}$$

which one also could obtain by assuming differentiability in (56) and applying the envelope theorem. The expression collapses to (8) when $\gamma=\delta=0$.

8.0.4 Construction of $\hat{S}_t$

Two practical problems face us when constructing a proxy for S. First, (3) will sometimes lead S to be negative. That is, if we use (**??**) as a proxy for λk, the resulting S will become negative. To prevent this from happening, we change (3) to

$$S' = \max(0, \lambda k + S - X). \tag{58}$$

Second, we face a units-conversion problem. What we measure, though, is not S but its proxy, $\hat{S}$, which we shall assume obeys the equation

$$\hat{S} = \theta S = \theta\lambda k \equiv \text{ NEW PATENTS \& TRADEMARKS},$$

Since S is measured in consumption units, θ is the number of $\hat{S}$ units per unit of consumption. (Our measures of X and k are already in consumption units). Substituting for S into (58),

$$\frac{1}{\theta}\hat{S}' = \max\left(0, \lambda k + \frac{\hat{S}}{\theta} - X\right), \text{ i.e., } \hat{S}' = \max\left(0, \theta\lambda k + \hat{S} - \theta X\right),$$

i.e.,

$$\frac{\hat{S}'}{k'}\left(1 - \delta + x\right) = \max\left(0, \theta\lambda + \hat{s} - \theta x\right),$$

where $\hat{s} = \frac{\hat{S}}{k}$. Therefore the law of motion for $\hat{s}$ is

$$\hat{s}' = \frac{\max\left(0, \theta\lambda + \hat{s} - \theta x\right)}{1 - \delta + x},$$

i.e.,

$$\hat{s}' = \frac{\max\left(0, \frac{\text{NEW PATENTS \&TRADEMARKS}}{\text{CAPITAL STOCK}} + \hat{s} - \theta x\right)}{1 - \delta + x},$$

or, dividing both sides by θ,

$$s' = \frac{\max\left(0, \frac{\text{NEW PATENTS \&TRADEMARKS}}{\theta\text{CAPITAL STOCK}} + s - x\right)}{1 - \delta + x}.$$