Methods for Robust Control

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Abstract

Robust control allows policymakers to formulate policies that guard against model misspecification. The principal tools used to solve robust control problems are state-space methods (see Hansen and Sargent, 2005, and Giordani and Söderlind, 2004). In this paper we show that the structural-form methods developed by Dennis (2005a) to solve control problems with rational expectations can also be applied to robust control problems, with the advantage that they bypass the task, often onerous, of having to express the reference model in a state-space form. Interestingly, state-space and structural-form methods do not necessarily return the same equilibria for robust control problems. We apply both state-space and structural solution methods to an empirical New Keynesian business cycle model and find that the differences between the methods are both qualitatively and quantitatively important. In particular, with the structural-form solution methods the specification errors generally involve changes to the conditional variances in addition to the conditional means of the shock processes.

Keywords: Robust control, Misspecification, Optimal policy.

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1 Introduction

The precision with which economic models can be expressed mathematically belies the fact that they cannot claim to be anything more than approximations to an unknown, and possibly unknowable, data-generating process. This unfortunate reality means that economic decisions are inevitably made in situations where important aspects of the environment are cloaked, hidden behind a cloud of uncertainty. While such uncertainty is hardly welcome, it need not render decisionmakers powerless, as its effects can in principle be mitigated through the application of robust control methods. Robust control provides a set of tools to assist decisionmakers confronting uncertainty who are either unable or unwilling to specify a probability distribution over possible specification errors. While robust control methods hold special appeal to policymakers, such as central banks, for whom models often play an explicit role in the decisionmaking process, they also allow private agents to express concern, or pessimism, when forming expectations.

The theory establishing that robust control methods can be applied to economic problems has been developed largely in a series of contributions by Hansen and Sargent, contributions that are well summarized in Hansen and Sargent (2005). Among other things, Hansen and Sargent show how to set up and solve discounted robust control problems, and they develop methods to solve for robust policies in backward-looking models and in forward-looking models with commitment. Giordani and Söderlind (2004) extend these methods to forward-looking models with discretion and to simple rules.

A critical component in the application of robust control is the reference model. A reference model is a structural model, possibly arrived at through some (nonmodeled) learning process, that is thought to be a good approximation to the underlying data-generating process. The methods developed in Hansen and Sargent (2005) and Giordani and Söderlind (2004) require that this reference model be written in a state-space form, following the literature on traditional (non robust) optimal control. As discussed in Dennis (2005a), while state-space methods allow models to be expressed in a form that contains only first-order dynamics, they also have drawbacks. In particular, many models cannot be expressed easily in a state-space form, especially medium- to large-scale models for which the necessary manipulations are often prohibitive. For robust control problems, the state-space formulation has an additional important implication in that the policymaker and the fictitious "evil agent" are not treated symmetrically. Specifically, the planner's decisions can affect current period outcomes both directly and through private sector expectations, while the evil agent's decisions can only affect current period outcomes through private sector expectations. As we show in this paper, this feature of the traditional robust control setup means that the evil agent will introduce specification errors by changing the conditional means of the shock processes, but not their conditional volatility.

In this paper we develop an alternative set of tools to solve robust control problems, tools based on the solution methods developed by Dennis (2005a) that have the advantage that they do not require the reference model to be written in a state-space form. Instead they allow the reference model to be written in structural form, which is more flexible and generally much easier to attain than is a state-space form. As we show, the structural form also allows us to treat the policymaker and the evil agent symmetrically, giving rise to the result that the evil agent will optimally choose to change the conditional volatility of the shocks in addition to their conditional means.

To illustrate how the structural-form solution methods work and to show how they differ from state-space methods, we apply both methods to the empirical New Keynesian business cycle model estimated by Rudebusch (2002a). We find that the differences between the state-space and the structural-form methods have effects on the economy and implications for monetary policy that are both qualitatively and quantitatively important.

The paper is structured as follows. Section 2 describes the standard state-space method to applying robust control and documents the properties of the resulting equilibria. Section 3 describes how robust control problems can be formulated and solved when the model is kept in a structural form rather than expressed in a state-space form. This section argues that the equilibria obtained using the structural-form methods are not necessarily the same as those obtained using the state-space form and shows that the differences have important behavioral implications. Section 4 shows why the two methods can give different solutions, showing that for robust control problems the state-space methods restrict the state variables in a way that is not necessarily desirable. In Section 5 we apply the two methods to an empirical New Keynesian model of the U.S. economy. Section 6 concludes.

2 Robust control using state-space methods

Hansen and Sargent (2005) describe how robust control methods, which allow for model uncertainty, can be used to design robustly optimal policies. They show that robust control problems can be cast in a form that allows them to be solved using methods that are standard in situations where expectations are rational. In particular, Hansen and Sargent (2003) show that state-space methods, such as those developed by Oudiz and Sachs (1985), Currie and Levine (1985, 1993), and Backus and Driffill (1986), can be applied to robust control problems to obtain Ramsey, or commitment, equilibria.

When solving robust control problems there are generally two distinct equilibria that are of interest. The first is the "worst-case" equilibrium, which is the equilibrium that pertains when the policymaker and private agents design policy and form expectations based on the worst-case misspecification and the worst-case misspecification is realized. The second is the "approximating" equilibrium, which is the equilibrium that pertains when the policymaker and private agents design policy and form expectations based on the worst-case misspecification, but the reference model transpires to be specified correctly. In this section we outline how state-space methods can be used to obtain these two equilibria, setting the scene for the structural-form analysis that follows. We focus on commitment, leaving the solution under discretion to Appendix A (see also Giordani and Söderlind, 2004).

2.1 Constraints and objectives

According to the state-space formulation, the economic environment is one in which the behavior of an $n \times 1$ vector of endogenous variables, $\mathbf{z}_t$, consisting of n_1 predetermined variables, $\mathbf{z}_{1t}$, and n_2 ($n_2 = n - n_1$) non-predetermined variables, $\mathbf{z}_{2t}$, are governed by the reference model

$$\mathbf{z}_{1t+1} = \mathbf{A}_{11}\mathbf{z}_{1t} + \mathbf{A}_{12}\mathbf{z}_{2t} + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\boldsymbol{\varepsilon}_{1t+1}, \tag{1}$$

$$\mathrm{E}_t\mathbf{z}_{2t+1} = \mathbf{A}_{21}\mathbf{z}_{1t} + \mathbf{A}_{22}\mathbf{z}_{2t} + \mathbf{B}_2\mathbf{u}_t, \tag{2}$$

where $\mathbf{u}_t$ is a $p \times 1$ vector of control variables, $\boldsymbol{\varepsilon}_{1t} \sim iid\,[\mathbf{0}, \mathbf{I}_s]$ is an $s \times 1$, $s \leq n_1$, vector of white-noise innovations, and E_t is the mathematical expectations operator conditional upon information available up to and including period t. The reference model is the model that private agents and the policymaker believe most accurately describes the data generating process. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ contain structural parameters and are conformable with $\mathbf{z}_{1t}$, $\mathbf{z}_{2t}$, and $\mathbf{u}_t$ as necessary. The matrix $\mathbf{C}_1$ is determined to ensure that $\boldsymbol{\varepsilon}_{1t}$ has the identity matrix as its variance-covariance matrix.

The policymaker's problem is to choose a sequence for its control variables, $\{\mathbf{u}_t\}_0^\infty$, to minimize the objective function

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + 2\mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t\right], \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor. The weighting matrices, $\mathbf{W}$, $\mathbf{U}$, and $\mathbf{R}$ reflect the policymaker's preferences; $\mathbf{W}$ and $\mathbf{R}$ are assumed to be positive semidefinite and positive definite, respectively.

Acknowledging that their reference model may be misspecified, private agents and the policymaker surround their reference model with a class of models of the form

$$\mathbf{z}_{1t+1} = \mathbf{A}_{11}\mathbf{z}_{1t} + \mathbf{A}_{12}\mathbf{z}_{2t} + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\left(\mathbf{v}_{t+1} + \boldsymbol{\varepsilon}_{1t+1}\right), \tag{4}$$

$$\mathrm{E}_t\mathbf{z}_{2t+1} = \mathbf{A}_{21}\mathbf{z}_{1t} + \mathbf{A}_{22}\mathbf{z}_{2t} + \mathbf{B}_2\mathbf{u}_t, \tag{5}$$

where $\mathbf{v}_{t+1}$ is a vector of specification errors, to arrive at a "distorted" model. The specification errors are intertemporally constrained to satisfy

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} \mathbf{v}_{t+1}' \mathbf{v}_{t+1} \leq \eta, \tag{6}$$

where $\eta \in [0, \bar{\eta})$ represents the "budget" for misspecification.

Because private agents form expectations that are "rational" according to the distorted model, the non-predetermined variables and their expected values are linked according to $\mathbf{z}_{2t+1} = \mathrm{E}_t\mathbf{z}_{2t+1} + \boldsymbol{\varepsilon}_{2t+1}$, where $\boldsymbol{\varepsilon}_{2t}$ is a martingale difference sequence, and the distorted model can be written as

$$\mathbf{z}_{1t+1} = \mathbf{A}_{11}\mathbf{z}_{1t} + \mathbf{A}_{12}\mathbf{z}_{2t} + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\left(\mathbf{v}_{t+1} + \boldsymbol{\varepsilon}_{1t+1}\right), \tag{7}$$

$$\mathbf{z}_{2t+1} = \mathbf{A}_{21}\mathbf{z}_{1t} + \mathbf{A}_{22}\mathbf{z}_{2t} + \mathbf{B}_2\mathbf{u}_t + \boldsymbol{\varepsilon}_{2t+1}, \tag{8}$$

or, more compactly and in obvious notation, as

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t + \mathbf{C}\mathbf{v}_{t+1} + \widetilde{\mathbf{C}}\boldsymbol{\varepsilon}_{t+1}. \tag{9}$$

To guard against the worst case misspecification, the policymaker formulates policy subject to the distorted model with the view that the misspecification will be as damaging as possible. Private sector agents form expectations with the same view. The fear that the misspecification will be as damaging as possible is operationalized through the metaphor that $\mathbf{v}_{t+1}$ is chosen by an evil agent whose objectives are diametrically opposed to those of the policymaker.[1] Hansen and Sargent (2001) show that the constraint problem in which equation (3) is minimized with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximized with respect to $\{\mathbf{v}_t\}_1^\infty$, subject to equations (9) and (6), can be recast in terms of an equivalent multiplier problem, whereby

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t'\mathbf{W}\mathbf{z}_t + 2\mathbf{z}_t'\mathbf{U}\mathbf{u}_t + \mathbf{u}_t'\mathbf{R}\mathbf{u}_t - \theta\mathbf{v}_{t+1}'\mathbf{v}_{t+1}\right] \tag{10}$$

is minimized with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximized with respect to $\{\mathbf{v}_t\}_1^\infty$, subject to

[1] Note that $\mathbf{v}_{t+1}$ is dated at $t+1$ although it is chosen at t. This convention is due to the fact that the specification errors are disguised by the innovations occurring at $t+1$.

equation (9). The parameter $\theta \in [\underline{\theta}, \infty)$ is a shadow price that is inversely related to the budget for misspecification η, Specifically, as η approaches zero, θ approaches infinity.

2.2 Robust policymaking with commitment

In the commitment solution both the policymaker and the evil agent are assumed to commit to a policy strategy and not succumb to incentives to renege on that strategy. Employing the definitions

$$\widetilde{\mathbf{u}}_t \equiv \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_{t+1} \end{bmatrix}, \quad \widetilde{\mathbf{B}} \equiv \begin{bmatrix} \mathbf{B} & \mathbf{C}_1 \end{bmatrix}, \tag{11}$$

$$\widetilde{\mathbf{U}} \equiv \begin{bmatrix} \mathbf{U} & \mathbf{0} \end{bmatrix}, \quad \widetilde{\mathbf{R}} \equiv \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & -\theta \mathbf{I} \end{bmatrix}, \tag{12}$$

the optimization problem can be written as

$$\min_{\{\mathbf{u}_t\}} \max_{\{\mathbf{v}_{t+1}\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t' \mathbf{W} \mathbf{z}_t + 2\mathbf{z}_t' \widetilde{\mathbf{U}} \widetilde{\mathbf{u}}_t + \widetilde{\mathbf{u}}_t' \widetilde{\mathbf{R}} \widetilde{\mathbf{u}}_t \right], \tag{13}$$

subject to

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \widetilde{\mathbf{B}} \widetilde{\mathbf{u}}_t + \widetilde{\mathbf{C}} \boldsymbol{\varepsilon}_{t+1}, \tag{14}$$

which, because the first-order conditions for a maximum are the same as those for a minimum, has a form that can be solved using the methods developed by Backus and Driffill (1986). Those methods involve formulating the optimization problem as a dynamic program. Recognizing that the problem is linear-quadratic, the value function has the form $V(\mathbf{z}_t) = \mathbf{z}_t' \mathbf{V} \mathbf{z}_t + d$ and the dynamic program can be written as

$$\mathbf{z}_t' \mathbf{V} \mathbf{z}_t + d \equiv \min_{\mathbf{u}_t} \max_{\mathbf{v}_{t+1}} \left[\mathbf{z}_t' \mathbf{W} \mathbf{z}_t + 2\mathbf{z}_t' \widetilde{\mathbf{U}} \widetilde{\mathbf{u}}_t + \widetilde{\mathbf{u}}_t' \widetilde{\mathbf{R}} \widetilde{\mathbf{u}}_t + \beta \mathrm{E}_t \left(\mathbf{z}_{t+1}' \mathbf{V} \mathbf{z}_{t+1} + d \right) \right]. \tag{15}$$

It is well known that the solution to this optimization problem takes the form

$$\begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_{t+1} \end{bmatrix} = -\mathbf{F}\mathbf{T}^{-1} \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{p}_{2t} \end{bmatrix}, \tag{16}$$

$$\mathbf{z}_{2t} = \begin{bmatrix} \mathbf{V}_{22}^{-1} \mathbf{V}_{21} & \mathbf{V}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{p}_{2t} \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} \mathbf{z}_{1t+1} \\ \mathbf{p}_{2t+1} \end{bmatrix} = \mathbf{T}\left(\mathbf{A}-\widetilde{\mathbf{B}}\mathbf{F}\right)\mathbf{T}^{-1}\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{p}_{2t} \end{bmatrix} + \mathbf{C}\boldsymbol{\varepsilon}_{1t+1}, \tag{18}$$

where $\mathbf{p}_{2t}$ is an $n_2 \times 1$ vector of shadow prices associated with the non-predetermined variables, $\mathbf{z}_{2t}$. The matrix $\mathbf{T}$ provides a mapping between the state variables, $\mathbf{z}_{1t}$ and $\mathbf{p}_{2t}$, and $\mathbf{z}_t$ and is given by

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}, \tag{19}$$

where $\mathbf{V}_{21}$ and $\mathbf{V}_{22}$ are submatrices of $\mathbf{V}$. Finally, $\mathbf{V}$ and $\mathbf{F}$ are obtained by solving for the fix-point of

$$\mathbf{V} = \mathbf{W} - 2\widetilde{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widetilde{\mathbf{R}}\mathbf{F} + \beta\left(\mathbf{A}-\widetilde{\mathbf{B}}\mathbf{F}\right)'\mathbf{V}\left(\mathbf{A}-\widetilde{\mathbf{B}}\mathbf{F}\right), \tag{20}$$

$$\mathbf{F} = \left(\widetilde{\mathbf{R}} + \beta\widetilde{\mathbf{B}}'\mathbf{V}\widetilde{\mathbf{B}}\right)^{-1}\left(\widetilde{\mathbf{U}}' + \beta\widetilde{\mathbf{B}}'\mathbf{V}\mathbf{A}\right). \tag{21}$$

When the worst case misspecification is realized, the economy behaves according to equations (16)–(18). While the worst case equilibrium is certainly interesting, it is also important to consider how the economy behaves when the reference model transpires to be specified correctly. Partitioning $\mathbf{F}$ into $[\ \mathbf{F}'_{\mathbf{u}}\ \ \mathbf{F}'_{\mathbf{v}}\ ]'$ where $\mathbf{F}_{\mathbf{u}}$ and $\mathbf{F}_{\mathbf{v}}$ are conformable with $\mathbf{u}_t$ and $\mathbf{v}_{t+1}$, respectively, Dennis (2005b) shows that the approximating equilibrium has the form

$$\mathbf{z}_{1t+1} = \left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}_{21} + \mathbf{B}_1\mathbf{F}^{\mathbf{u}}_{\mathbf{z}1}\right)\mathbf{z}_{1t} + \left(\mathbf{A}_{12}\mathbf{H}_{22} + \mathbf{B}_1\mathbf{F}^{\mathbf{u}}_{\mathbf{p}2}\right)\mathbf{p}_{2t} + \mathbf{C}_1\boldsymbol{\varepsilon}_{1t+1}, \tag{22}$$

$$\mathbf{p}_{2t+1} = \mathbf{M}_{21}\mathbf{z}_{1t} + \mathbf{M}_{22}\mathbf{p}_{2t}, \tag{23}$$

$$\mathbf{z}_{2t} = \mathbf{H}_{21}\mathbf{z}_{1t} + \mathbf{H}_{22}\mathbf{p}_{2t}, \tag{24}$$

$$\mathbf{u}_{t} = \mathbf{F}^{\mathbf{u}}_{\mathbf{z}1}\mathbf{z}_{1t} + \mathbf{F}^{\mathbf{u}}_{\mathbf{p}2}\mathbf{p}_{2t}, \tag{25}$$

where $\mathbf{H}_{21} \equiv \mathbf{V}_{22}^{-1}\mathbf{V}_{21}$, $\mathbf{H}_{22} \equiv \mathbf{V}_{22}^{-1}$, $\begin{bmatrix} \mathbf{F}^{\mathbf{u}}_{\mathbf{z}1} & \mathbf{F}^{\mathbf{u}}_{\mathbf{p}2} \end{bmatrix} \equiv -\mathbf{F}_{\mathbf{u}}\mathbf{T}^{-1}$, and

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \equiv \mathbf{T}\left(\mathbf{A}-\widetilde{\mathbf{B}}\mathbf{F}\right)\mathbf{T}^{-1}. \tag{26}$$

Interestingly, the worst-case equilibrium and the approximating equilibrium share certain features. For instance, the worst-case equilibrium and the approximating equilibrium differ only with respect to the law of motion for the predetermined variables and, as a consequence, following innovations to the system the initial-period responses of the predetermined variables are the same for the approximating equilibrium as for the worst-case equilibrium. But since the decision rules for $\mathbf{z}_{2t}$ and $\mathbf{u}_t$ are also the same for the two equilibria, it follows that the initial-period responses by the non-predetermined variables and by the policy variables are also the same. With respect to impulse response functions, differences between the approximating equilibrium and the worst-case equilibrium then only occur one period after innovations occur.

Furthermore, because the coefficient matrix on the innovations is $\mathbf{C}_1$, which scales the standard deviations of the innovations, it follows that adding noise to the innovations or changing their correlation structure is not part of the evil agent's strategy. Instead, the optimally designed misspecification has the effect of changing the law of motion for the predetermined variables. More precisely, since the specification errors enter only the stochastic component of $\mathbf{z}_{1t}$, the evil agent's strategy is to change the conditional means of the shock processes but not their conditional volatility. As shown in Appendix A, these relationships between the worst-case and the approximating equilibria also hold under discretion.

3 Robust control using structural-form methods

While state-space solution methods have many advantages, being generally compact and containing only first-order dynamics, they are not always convenient. In particular, problems can arise from the fact that it is often difficult, sometimes prohibitively so, to manipulate a model into a state-space form, making state-space methods better suited to small models. But policymakers often employ medium- to large-scale models, and for this reason alone it is desirable to be able to solve robust control problems without relying on state-space methods. In this regard, Dennis (2005a) has developed numerical methods that solve for optimal commitment policies and optimal discretionary policies in rational expectations models that allow the optimization constraints to be written in a structural form. These structural-form solution methods are easy to apply and offer considerable flexibility with regard to how the model is expressed.

One contribution of this section is to show that these structural-form methods can be readily applied to solve robust control problems. In fact, the advantages to using structural-form methods may extend somewhat further than convenience and flexibility. Leitemo and Söderström (2004, 2005) use a Lagrangian method—with the constraints in a structural form—to solve analytically for robustly optimal discretionary policies in closed- and open-economy models, respectively. They find that the evil agent's optimal strategy is to change the variances of the shocks, not their persistence, a strategy that differs from what the state-space methods outlined above would suggest.

In addition to illustrating how structural-form methods can be used to solve robust control problems numerically, we demonstrate that they need not generate the same worst case equilibrium as the state-space methods and explain why. We note that whereas with state-space methods the evil agent's strategy is to change the conditional means of the shocks, with structural-form methods the evil agent will generally choose to change both the conditional means and the variance/covariance structure of the shocks. As we show, these differences arise because the structural-form solution methods change slightly the nature of the game played between the agents in the model, accommodating a more general class of specification errors in the process. Finally, we outline how detection-error probabilities, essentially, the probability that an econometrician would make a model selection error, can be calculated given this more general class of specification errors.

3.1 Constraints and objectives

The basic model representation that Dennis (2005a) works with is the second-order structural form. Therefore, let the reference model be represented as

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathrm{E}_t\mathbf{y}_{t+1} + \mathbf{A}_3\mathbf{u}_t + \mathbf{A}_4\boldsymbol{\varepsilon}_t, \tag{27}$$

where $\mathbf{y}_t$ is an $n\times1$ vector of endogenous variables, $\mathbf{u}_t$ is a $p\times1$ vector of policy instruments, $\boldsymbol{\varepsilon}_t$ is an $s\times1$, $0<s\leq n$, vector of innovations, and $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ are matrices with dimensions conformable with $\mathbf{y}_t$, $\mathbf{u}_t$, and $\boldsymbol{\varepsilon}_t$ that contain the structural parameters. The matrix $\mathbf{A}_0$ is assumed to be nonsingular and the elements of $\mathbf{A}_4$ are determined to ensure that the shocks are distributed according to $\boldsymbol{\varepsilon}_t \sim iid\,[0,\mathbf{I}_s]$. The dating on the variables is such that any variable that enters $\mathbf{y}_{t-1}$ is known by the beginning of period t; by construction the variables in $\mathbf{y}_{t-1}$ are predetermined. Binder and Pesaran (1995) show that this second-order structural form encompasses an enormous class of (log-) linear macroeconomic models.

With the reference model written in second-order structural form, private agents and the policymaker acknowledge their concern for misspecification by surrounding their reference model with a class of models of the form

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathrm{E}_t\mathbf{y}_{t+1} + \mathbf{A}_3\mathbf{u}_t + \mathbf{A}_4\left(\mathbf{v}_t+\boldsymbol{\varepsilon}_t\right), \tag{28}$$

where $\mathbf{v}_t$ is a vector containing specification errors and equation (28) represents the "distorted" model. Just as earlier, the specification errors are intertemporally constrained to satisfy:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \mathbf{v}_t' \mathbf{v}_t \leq \omega, \tag{29}$$

where $\omega \in [0, \overline{\omega})$ represents the evil agent's total budget for misspecification.

The policy objective function is taken to be

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t' \mathbf{W} \mathbf{y}_t + \mathbf{u}_t' \mathbf{Q} \mathbf{u}_t \right], \tag{30}$$

where $\mathbf{W}$ $(n \times n)$ and $\mathbf{Q}$ $(p \times p)$ are matrices containing policy weights and are symmetric positive semidefinite, and symmetric positive definite, respectively. Penalty terms on the interaction between $\mathbf{y}_t$ and $\mathbf{u}_t$ could be included, but are unnecessary because such terms can be accommodated through a suitable construction of $\mathbf{y}_t$, reflecting the greater flexibility offered by the structural form.

Analogous to the state-space approach, the problem of minimizing equation (30) with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximizing with respect to $\{\mathbf{v}_t\}_0^\infty$ subject to equations (28) and (29) can be replaced with an equivalent multiplier problem in which

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t' \mathbf{W} \mathbf{y}_t + \mathbf{u}_t' \mathbf{Q} \mathbf{u}_t - \phi \mathbf{v}_t' \mathbf{v}_t \right], \tag{31}$$

is minimized with respect to $\{\mathbf{u}_t\}_0^\infty$ and maximized with respect to $\{\mathbf{v}_t\}_0^\infty$, subject to equation (28). The multiplier $\phi \in \left[\underline{\phi}, \infty\right)$ is inversely related to the budget for misspecification, ω. This method of formulating the robust control problem with the reference model and the distorted model in structural form parallels Hansen and Sargent (2005) closely. Nevertheless, we distinguish between ω and η and between ϕ and θ to acknowledge that ϕ and θ, while they are both shadow prices, need not share the same interpretation and that $\overline{\omega}$ and $\overline{\eta}$ need not take on the same value. A generalization that we do not pursue here, but that is discussed and implemented in Leitemo and Söderström (2005), is to assign separate budgets to each of the evil agent's controls.

3.2 Robust policymaking with commitment

To solve the robust control problem with commitment when the constraints are in second-order structural form the optimization problem is formulated using the Lagrangian

$$L = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t' \mathbf{W} \mathbf{y}_t + \widetilde{\mathbf{u}}_t' \widetilde{\mathbf{Q}} \widetilde{\mathbf{u}}_t + 2 \boldsymbol{\lambda}_t' \left(\mathbf{A}_0 \mathbf{y}_t - \mathbf{A}_1 \mathbf{y}_{t-1} - \mathbf{A}_2 \mathbf{y}_{t+1} - \widetilde{\mathbf{A}}_3 \widetilde{\mathbf{u}}_t - \rho_t \right) \right], \tag{32}$$

where

$$\widetilde{\mathbf{Q}} \equiv \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & -\phi\mathbf{I} \end{bmatrix}, \quad \widetilde{\mathbf{A}}_3 \equiv \begin{bmatrix} \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \widetilde{\mathbf{u}}_t \equiv \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix}, \tag{33}$$

and $\rho_t \equiv \mathbf{A}_4\boldsymbol{\varepsilon}_t - \mathbf{A}_2\boldsymbol{\varepsilon}_t^y$, with $\boldsymbol{\varepsilon}_t^y \equiv (\mathbf{y}_{t+1} - \mathrm{E}_t\mathbf{y}_{t+1})$. The first-order conditions with respect to $\widetilde{\mathbf{u}}_t$, $\boldsymbol{\lambda}_t$, and $\mathbf{y}_t$, respectively, can be written as

$$\frac{\partial L}{\partial \widetilde{\mathbf{u}}_t} = \widetilde{\mathbf{Q}}\widetilde{\mathbf{u}}_t - \widetilde{\mathbf{A}}_3'\boldsymbol{\lambda}_t = \mathbf{0}, \; t \geq t_0, \tag{34}$$

$$\frac{\partial L}{\partial \boldsymbol{\lambda}_t} = \mathbf{A}_0\mathbf{y}_t - \mathbf{A}_1\mathbf{y}_{t-1} - \mathbf{A}_2\mathrm{E}_t\mathbf{y}_{t+1} - \widetilde{\mathbf{A}}_3\widetilde{\mathbf{u}}_t - \mathbf{A}_4\boldsymbol{\varepsilon}_t = \mathbf{0}, \; t \geq t_0, \tag{35}$$

$$\frac{\partial L}{\partial \mathbf{y}_t} = \mathbf{W}\mathbf{y}_t + \mathbf{A}_0'\boldsymbol{\lambda}_t - \beta^{-1}\mathbf{A}_2'\boldsymbol{\lambda}_{t-1} - \beta\mathbf{A}_1'\mathrm{E}_t\boldsymbol{\lambda}_{t+1} = \mathbf{0}, \; t \geq t_0, \tag{36}$$

with the initial condition that $\boldsymbol{\lambda}_{t-1} = \mathbf{0}$.[2] Equations (34)–(36) describe a standard system of expectational equations, in which the expectations are formed rationally from the perspective of the distorted model and can be solved in a variety of ways. However this system is solved, the solution can be written as

$$\begin{bmatrix} \boldsymbol{\lambda}_t \\ \mathbf{y}_t \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\boldsymbol{\lambda}\boldsymbol{\lambda}} & \mathbf{H}_{\boldsymbol{\lambda}\mathbf{y}} \\ \mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}} & \mathbf{H}_{\mathbf{y}\mathbf{y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_{t-1} \\ \mathbf{y}_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_{\boldsymbol{\lambda}\boldsymbol{\varepsilon}} \\ \mathbf{G}_{\mathbf{y}\boldsymbol{\varepsilon}} \end{bmatrix} \boldsymbol{\varepsilon}_t, \tag{37}$$

$$\widetilde{\mathbf{u}}_t = \begin{bmatrix} \mathbf{F}_{\boldsymbol{\lambda}} & \mathbf{F}_{\mathbf{y}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_{t-1} \\ \mathbf{y}_{t-1} \end{bmatrix} + \mathbf{F}_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t. \tag{38}$$

Equations (37) and (38) describe how the economy behaves in the worst-case equilibrium.

Given the worst-case equilibrium, the approximating equilibrium, which is the equilibrium that pertains when the reference model is actually correctly specified, is

$$\boldsymbol{\lambda}_t = \mathbf{H}_{\boldsymbol{\lambda}\boldsymbol{\lambda}}\boldsymbol{\lambda}_{t-1} + \mathbf{H}_{\boldsymbol{\lambda}\mathbf{y}}\mathbf{y}_{t-1} + \mathbf{G}_{\boldsymbol{\lambda}\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t, \tag{39}$$

$$\mathbf{u}_t = \mathbf{F}^{\mathbf{u}}_{\boldsymbol{\lambda}}\boldsymbol{\lambda}_{t-1} + \mathbf{F}^{\mathbf{u}}_{\mathbf{y}}\mathbf{y}_{t-1} + \mathbf{F}^{\mathbf{u}}_{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_t, \tag{40}$$

$$\begin{aligned} \mathbf{y}_t = {} & \mathbf{A}_0^{-1}\left[\mathbf{A}_1 + \mathbf{A}_2\left(\mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}}\mathbf{H}_{\boldsymbol{\lambda}\mathbf{y}} + \mathbf{H}_{\mathbf{y}\mathbf{y}}\mathbf{H}_{\mathbf{y}\mathbf{y}}\right) + \mathbf{A}_3\mathbf{F}^{\mathbf{u}}_{\mathbf{y}}\right]\mathbf{y}_{t-1} \\ & + \mathbf{A}_0^{-1}\left[\mathbf{A}_2\left(\mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}}\mathbf{H}_{\boldsymbol{\lambda}\boldsymbol{\lambda}} + \mathbf{H}_{\mathbf{y}\mathbf{y}}\mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}}\right) + \mathbf{A}_3\mathbf{F}^{\mathbf{u}}_{\boldsymbol{\lambda}}\right]\boldsymbol{\lambda}_{t-1} \\ & + \mathbf{A}_0^{-1}\left[\mathbf{A}_4 + \mathbf{A}_2\left(\mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}}\mathbf{G}_{\boldsymbol{\lambda}\boldsymbol{\varepsilon}} + \mathbf{H}_{\mathbf{y}\mathbf{y}}\mathbf{G}_{\mathbf{y}\boldsymbol{\varepsilon}}\right) + \mathbf{A}_3\mathbf{F}^{\mathbf{u}}_{\boldsymbol{\varepsilon}}\right]\boldsymbol{\varepsilon}_t. \end{aligned} \tag{41}$$

Recall that for the state-space solution methods there were certain relationships between the worst-case equilibrium and the approximating equilibrium, relationships that

[2] This initial condition is not arbitrary; it emerges from the optimal program through the fact that all promises made prior to period 0 are ignored in period 0.

held for both commitment and discretion. Specifically, the evil agent's strategy involved changing the persistence properties of the shocks, but not the volatility of the innovations, which meant that the initial period responses of the predetermined variables, the non-predetermined variables, and the policy controls to innovations would be the same for the worst-case equilibrium and the approximating equilibrium. Using the structural-form solution methods described above, however, these relationships do not necessarily hold.

To see this, note that the contemporaneous response of $\mathbf{y}_t$ to $\boldsymbol{\varepsilon}_t$ is $\mathbf{G}_{\mathbf{y}\boldsymbol{\varepsilon}}$ in the worst-case equilibrium (see equation (37)) and $\mathbf{A}_0^{-1}\left[\mathbf{A}_4 + \mathbf{A}_2\left(\mathbf{H}_{\mathbf{y}\boldsymbol{\lambda}}\mathbf{G}_{\boldsymbol{\lambda}\boldsymbol{\varepsilon}} + \mathbf{H}_{\mathbf{yy}}\mathbf{G}_{\mathbf{y}\boldsymbol{\varepsilon}}\right) + \mathbf{A}_3\mathbf{F}_{\boldsymbol{\varepsilon}}^{\mathbf{u}}\right]$ in the approximating equilibrium (see equation (41)). When these structural-form methods are employed the evil agent's strategy may well involve a change to the variance-covariance matrix of the innovations as well as a change to the conditional means of the shock processes. It follows that the initial period responses by the endogenous variables, and hence also by the policy controls, to innovations may also differ between the worst-case and the approximating equilibria.

3.3 Robust policymaking with discretion

In the discretionary environment the optimization problem remains to

$$\min_{\{\mathbf{u}_t\}_0^\infty}\max_{\{\mathbf{v}_t\}_0^\infty} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{y}_t'\mathbf{W}\mathbf{y}_t + \widetilde{\mathbf{u}}_t'\widetilde{\mathbf{Q}}\widetilde{\mathbf{u}}_t\right] \tag{42}$$

subject to

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathrm{E}_t\mathbf{y}_{t+1} + \widetilde{\mathbf{A}}_3\widetilde{\mathbf{u}}_t + \mathbf{A}_4\boldsymbol{\varepsilon}_t, \tag{43}$$

but, of course, neither the policymaker nor the evil agent can commit. The policymaker and the evil agent are Stackelberg leaders with respect to their future selves, but play a Cournot game between themselves. The problem described by equations (42) and (43) conforms to the class of problems studied and solved by Dennis (2005a), where it is shown that the solution takes the form

$$\mathbf{y}_t = \mathbf{H}\mathbf{y}_{t-1} + \mathbf{G}\boldsymbol{\varepsilon}_t, \tag{44}$$

$$\widetilde{\mathbf{u}}_t = \mathbf{F}_1\mathbf{y}_{t-1} + \mathbf{F}_2\boldsymbol{\varepsilon}_t. \tag{45}$$

The matrices $\mathbf{H}$, $\mathbf{G}$, $\mathbf{F}_1$, and $\mathbf{F}_2$ that govern the solution are arrived at through an iterative procedure. The first step involves conjecturing values for $\mathbf{H}$ and $\mathbf{F}_1$ and using these to solve for the matrix $\mathbf{D}$ and the fix-point $\mathbf{P}$ according to

$$\mathbf{D} \equiv \mathbf{A}_0 - \mathbf{A}_2\mathbf{H}, \tag{46}$$

$$\mathbf{P} \equiv \mathbf{W} + \beta\mathbf{F}_1'\widetilde{\mathbf{Q}}\mathbf{F}_1 + \beta\mathbf{H}'\mathbf{P}\mathbf{H}. \tag{47}$$

Next, the values for $\mathbf{D}$ and $\mathbf{P}$ that solve equations (46) and (47) are used together with the conjectured values for $\mathbf{H}$ and $\mathbf{F}_1$ to update $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{H}$, and $\mathbf{G}$ according to

$$\mathbf{F}_1 = -\left(\widetilde{\mathbf{Q}} + \widetilde{\mathbf{A}}_3'\mathbf{D}^{-1}\mathbf{P}\mathbf{D}^{-1}\widetilde{\mathbf{A}}_3\right)^{-1}\widetilde{\mathbf{A}}_3'\mathbf{D}'^{-1}\mathbf{P}\mathbf{D}^{-1}\mathbf{A}_1, \tag{48}$$

$$\mathbf{F}_2 = -\left(\widetilde{\mathbf{Q}} + \widetilde{\mathbf{A}}_3'\mathbf{D}'^{-1}\mathbf{P}\mathbf{D}^{-1}\widetilde{\mathbf{A}}_3\right)^{-1}\widetilde{\mathbf{A}}_3'\mathbf{D}'^{-1}\mathbf{P}\mathbf{D}^{-1}\mathbf{A}_4, \tag{49}$$

$$\mathbf{H} = \mathbf{D}^{-1}\left(\mathbf{A}_1 + \widetilde{\mathbf{A}}_3\mathbf{F}_1\right), \tag{50}$$

$$\mathbf{G} = \mathbf{D}^{-1}\left(\mathbf{A}_4 + \widetilde{\mathbf{A}}_3\mathbf{F}_2\right). \tag{51}$$

From equations (48) - (51), updates of $\mathbf{D}$ and the fix-point $\mathbf{P}$ are generated, which in turn give rise to updated values for $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{H}$, and $\mathbf{G}$. This iterative procedure continues until a fix-point in which $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{H}$, $\mathbf{G}$, and $\mathbf{P}$ no longer change with successive iterations is obtained.

Equations (44) and (45) govern the economy's behavior in the worst-case equilibrium. From this worst-case equilibrium, the approximating equilibrium can be easily constructed; it is given by

$$\mathbf{y}_t = \mathbf{A}_0^{-1}\left[\left(\mathbf{A}_1 + \mathbf{A}_2\mathbf{H}\mathbf{H} + \mathbf{A}_3\mathbf{F}_1^{\mathbf{u}}\right)\mathbf{y}_{t-1} + \left(\mathbf{A}_4 + \mathbf{A}_2\mathbf{H}\mathbf{G} + \mathbf{A}_3\mathbf{F}_2^{\mathbf{u}}\right)\boldsymbol{\varepsilon}_t\right], \tag{52}$$

$$\mathbf{u}_t = \mathbf{F}_1^{\mathbf{u}}\mathbf{y}_{t-1} + \mathbf{F}_2^{\mathbf{u}}\boldsymbol{\varepsilon}_t, \tag{53}$$

where equation (52) exploits the fact that $\mathbf{A}_0$ has full rank.

As one might expect, in the discretionary solution, just as in the commitment solution discussed above, the evil agent's strategy will generally involve changing both the persistence properties of the shocks and the variance-covariance matrix of the innovations. To see this, observe from equations (44) and (52) that the coefficient matrices on the innovations, $\mathbf{G}$, and $\mathbf{A}_4 + \mathbf{A}_2\mathbf{H}\mathbf{G} + \mathbf{A}_3\mathbf{F}_2^{\mathbf{u}}$, respectively, are not necessarily equal.

3.4 Detection-error probabilities

Anderson, Hansen, and Sargent (2003) describe the concept of a detection-error probability and introduce it as a tool for calibrating ϕ, the multiplier on the misspecification constraint, which would otherwise be a free parameter. A detection-error probability is the probability that an econometrician observing equilibrium outcomes would make an incorrect inference about whether the approximating equilibrium or the worst-case equilibrium generated the data. The intuitive connection between ϕ and the probability of making a detection error is that when ϕ is small, greater differences between the distorted model and the reference model (more severe misspecifications) can arise, which are more easily detected.

Let A and B denote two models; with a prior that assigns equal weight to each model, Hansen, Sargent, and Wang (2002) show that detection-error probabilities are calculated according to

$$p(\phi) = \frac{\text{prob}(A|B) + \text{prob}(B|A)}{2}, \tag{54}$$

where prob$(A|B)$ (prob$(B|A)$) represents the probability that the econometrician erroneously chooses model A (model B) when in fact model B (model A) generated the data. Let model A denote the approximating model and model B denote the worst-case model, then any sequence of specification errors that satisfies equation (29) will be at least as difficult to distinguish from the approximating model as is a sequence that satisfies equation (29) with equality. As such, $p(\phi)$ represents a lower bound on the probability of making a detection error.

To calculate a detection-error probability we require a description of how the econometrician goes about choosing one model over another. Hansen, Sargent, and Wang (2002) assume that this model selection is based on the likelihood ratio principle. Let $\{\mathbf{z}_t^B\}_1^T$ denote a finite sequence of economic outcomes generated according to the worst-case equilibrium, model B, and let L_{AB} and L_{BB} denote the likelihood associated with models A and B, respectively, then the econometrician chooses model A over model B if $\log(L_{BB}^n/L_{AB}^n) < 0$. Generating M independent sequences $\{\mathbf{z}_t^B\}_1^T$, prob$(A|B)$ can be calculated according to

$$\text{prob}(A|B) \approx \frac{1}{M}\sum_{m=1}^{M} \text{I}\left[\log\left(\frac{L_{BB}^m}{L_{AB}^m}\right) < 0\right], \tag{55}$$

where $\text{I}[\log(L_{BB}^m/L_{AB}^m) < 0]$ is the indicator function that equals one when its argument is satisfied and equals zero otherwise; prob$(B|A)$ is calculated analogously using draws generated from the approximating model. The likelihood function that is generally used to calculate prob$(A|B)$ and prob$(B|A)$ assumes that the innovations are normally distributed.

While the theory of detection does not require that the evil agent not distort the volatility of the innovations, existing methods to calculate detection-error probabilities do (see Hansen, Sargent, and Wang, 2002, for example). Here we show how to calculate detection-error probabilities while accounting for the distortions to both the conditional means and the conditional volatilities of the shocks. Let

$$\mathbf{z}_t = \mathbf{H}_A \mathbf{z}_{t-1} + \mathbf{G}_A \boldsymbol{\varepsilon}_t, \tag{56}$$

$$\mathbf{z}_t = \mathbf{H}_B \mathbf{z}_{t-1} + \mathbf{G}_B \boldsymbol{\varepsilon}_t \tag{57}$$

govern equilibrium outcomes under the approximating equilibrium and the worst-case equilibrium, respectively. With discretion $\mathbf{z}_t \equiv \mathbf{y}_t$ while with commitment $\mathbf{z}_t \equiv [\ \boldsymbol{\lambda}_t' \quad \mathbf{y}_t'\ ]'$. When $\mathbf{G}_A \neq \mathbf{G}_B$, to calculate $p(\phi)$ we must first allow for the stochastic singularity that generally characterizes equilibrium and second account appropriately for the Jacobian of transformation that enters the likelihood function. Using the QR decomposition we decompose $\mathbf{G}_A$ according to $\mathbf{G}_A = \mathbf{Q}_A \mathbf{R}_A$ and $\mathbf{G}_B$ according to $\mathbf{G}_B = \mathbf{Q}_B \mathbf{R}_B$. By construction, $\mathbf{Q}_A$ and $\mathbf{Q}_B$ are orthogonal matrices ($\mathbf{Q}_A' \mathbf{Q}_A = \mathbf{Q}_B' \mathbf{Q}_B = \mathbf{I}_s$) and $\mathbf{R}_A$ and $\mathbf{R}_B$ are upper triangular. Let

$$\widehat{\boldsymbol{\varepsilon}}_t^{i|j} = \mathbf{R}_i^{-1} \mathbf{Q}_i' \left(\mathbf{z}_t^j - \mathbf{H}_i \mathbf{z}_{t-1}^j \right), \quad \{i, j\} \in \{A, B\} \tag{58}$$

represent the inferred innovations in period t when model i is fitted to data $\{\mathbf{z}_t^j\}_1^T$ that are generated according to model j and let $\widehat{\boldsymbol{\Sigma}}^{i|j}$ be the associated estimates of the innovation variance-covariance matrices. Then

$$\log\left(\frac{L_{AA}}{L_{BA}}\right) = \log\left|\mathbf{R}_A^{-1}\right| - \log\left|\mathbf{R}_B^{-1}\right| + \frac{1}{2}\mathrm{tr}\left(\widehat{\boldsymbol{\Sigma}}^{B|A} - \widehat{\boldsymbol{\Sigma}}^{A|A}\right), \tag{59}$$

$$\log\left(\frac{L_{BB}}{L_{AB}}\right) = \log\left|\mathbf{R}_B^{-1}\right| - \log\left|\mathbf{R}_A^{-1}\right| + \frac{1}{2}\mathrm{tr}\left(\widehat{\boldsymbol{\Sigma}}^{A|B} - \widehat{\boldsymbol{\Sigma}}^{B|B}\right), \tag{60}$$

where "tr" is the trace operator.

When $\mathbf{G}_A = \mathbf{G}_B$ it follows that $\mathbf{R}_A = \mathbf{R}_B$ and the Jacobian of transformations associated with the various likelihoods cancel and play no role in the calculations, in which case equations (59) and (60) simplify to

$$\log\left(\frac{L_{AA}}{L_{BA}}\right) = \frac{1}{2}\mathrm{tr}\left(\widehat{\boldsymbol{\Sigma}}^{B|A} - \widehat{\boldsymbol{\Sigma}}^{A|A}\right), \tag{61}$$

$$\log\left(\frac{L_{BB}}{L_{AB}}\right) = \frac{1}{2}\mathrm{tr}\left(\widehat{\boldsymbol{\Sigma}}^{A|B} - \widehat{\boldsymbol{\Sigma}}^{B|B}\right), \tag{62}$$

which are equivalent to the expressions Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2005, chapter 8) employ. Given equations (59) and (60), equation (55) is used to estimate prob($A|B$) and (similarly) prob($B|A$), which are needed to construct the detection-error probability, as per equation (54). The multiplier, ϕ, is then determined by selecting a detection-error probability (or at least its lower bound) and inverting equation (54). Generally this inversion is performed numerically by constructing the mapping between ϕ and the detection-error probability, for a given sample size.

4 Comparing the solution methods

Sections 2 and 3 demonstrate that the solutions obtained for the worst-case equilibrium and the approximating equilibrium may depend on whether state-space methods or structural-form methods are used. Moreover, it should be clear that the differences between the two solution methods involve worst-case specification errors that are qualitatively different in important ways. For the structural-form solution methods, it is apparent that pessimistic agents are guarding against specification errors both to the conditional means of the shocks, which is the behavior Hansen and Sargent emphasize, and to the conditional variances/covariances of the shocks.

In an important sense, it is surprising that the solutions differ, as such differences do not arise when expectations are rational.[3] But since the methods may produce different equilibrium behavior, two important questions immediately present themselves: why do the differences arise, and are the differences quantitatively important? We defer the second question to Section 5, where both sets of tools are applied to a New Keynesian business cycle model. With regard to the first question, however, we show below that when the solutions differ they do so because the state-space formulation restricts the various decisionmakers in ways that the structural-form formulation does not. In effect, the two methods are solving closely related but not identical problems.

To see this point, consider the following simple example. Let the reference model that the policymaker and private agents share be

$$y_t = \alpha \mathrm{E}_t y_{t+1} + \gamma u_t + g_t, \tag{63}$$

$$g_t = \rho g_{t-1} + \sigma_\varepsilon \varepsilon_t, \tag{64}$$

where the parameters satisfy $\alpha \in (0,1)$, $\gamma \in (-\infty,\infty)$, $\rho \in (-1,1)$, and $\{\sigma_g, \sigma_\varepsilon\} \in (0,\infty)$, and where ε_t is a mean-zero white-noise process with standard deviation equal to σ_ε. Notice that ε_t is an exogenous variable, u_t is a decision variable, y_t, $\mathrm{E}_t y_{t+1}$, and g_t are non-predetermined variables, and g_{t-1} is a predetermined variable.[4]

To write equations (63) and (64) in state-space form the standard method would be to advance the timing on equation (64) one period and to make $\mathrm{E}_t y_{t+1}$ the subject of

[3] When expectations are rational, although the solutions obtained by state-space methods and structural-form methods are often presented in different forms, they are behaviorally equivalent.

[4] A variable is predetermined if its value next period can be forecasted perfectly using only information that is available today, i.e., a generic variable y_t is predetermined if $\mathrm{E}_t y_{t+1} = y_{t+1}$, see Engle, Hendry, and Richard (1983) and Blanchard and Kahn (1980).

equation (63), giving

$$\begin{bmatrix} g_{t+1} \\ \mathrm{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} g_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\gamma}{\alpha} \end{bmatrix} [u_t] + \begin{bmatrix} \sigma_\varepsilon \\ 0 \end{bmatrix} [\varepsilon_{t+1}]. \tag{65}$$

Adding the specification errors, the distorted model would then be

$$\begin{bmatrix} g_{t+1} \\ \mathrm{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ -\frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} g_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\gamma}{\alpha} \end{bmatrix} [u_t] + \begin{bmatrix} \sigma_\varepsilon \\ 0 \end{bmatrix} [v_{t+1} + \varepsilon_{t+1}]. \tag{66}$$

Notice that in equation (66) the shock g_t is a state variable, a variable that all agents take as given when forming decisions, even though it is not actually a predetermined variable.

In contrast, with the structural-form method once the model misspecifications are added to equation (64) the distorted model becomes

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} g_t \\ y_t \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} \mathrm{E}_t g_{t+1} \\ \mathrm{E}_t y_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} [u_t] + \begin{bmatrix} \sigma_\varepsilon \\ 0 \end{bmatrix} [v_t + \varepsilon_t]. \tag{67}$$

In equation (67) the state variables that agents take as given when forming decisions are g_{t-1} and ε_t. Thus, the key difference between the two representations is that in the structural-form representation the state variables are g_{t-1}, which is predetermined, and ε_t, which is exogenous, while in the state-space representation the state variable is g_t, which is non-predetermined.[5] Because the structural-form representation allows the evil agent to react separately to g_{t-1} and ε_t, if it so desires the evil agent can purposefully alter the realization of g_t, changing both the conditional mean of the shock and the variance of the innovation.[6]

Two final points are worth noting. First, although the structural-form representation does not restrict the state vector, and permits a wider class of specification errors as a consequence, because all agents in the model—not just the evil agent—have their behavior

[5] In the rational expectations context, although g_t is not actually predetermined, because its evolution is determined outside the system, unaffected by the actions of the agents in the economy, nothing is lost by making it a state variable and putting it in the predetermined block of the model.

[6] In the limit as the time between periods shortens and we approach continuous time, the distinction between g_{t-1} and g_t becomes inconsequential. It is in discrete-time models, then, that the state-space methods and the structural-form methods can generate different solutions. Hansen, Sargent, and Tallarini (1999) comment on a "small variance adjustment" that they associate with risk-sensitive preferences. They also note that its manifestation turns on the discreteness of time.

restricted it is not the case that relaxing this restriction necessarily allows the evil agent to do more damage for a given budget. Second, state-space forms (and structural forms) are not unique and for any given model a state-space representation that allows the evil agent to distort both the conditional mean and the conditional volatility of the shocks may be available.[7]

5 Robust policy in an empirical business cycle model

To illustrate the two solution approaches, we study the model estimated by Rudebusch (2002a), which is based on a standard New Keynesian model and contains two equations that, conditional upon the short-term interest rate, i_t, summarize the dynamics of inflation, π_t, and the output gap, y_t:

$$\pi_t = \mu_\pi \mathrm{E}_t \pi_{t+1} + (1-\mu_\pi)\,\pi_{t-1} + \alpha y_t + \varepsilon_{\pi,t}, \tag{68}$$

$$y_t = \mu_y \mathrm{E}_t y_{t+1} + \left(1-\mu_y\right) y_{t-1} - \beta\left[i_t - \mathrm{E}_t \pi_{t+1}\right] + \varepsilon_{y,t}. \tag{69}$$

Equation (68) is a "New Keynesian Phillips curve" derived from the optimal price-setting behavior of firms acting under monopolistic competition, but facing price rigidities, typically modeled following Calvo (1983). The presence of lagged inflation and the "supply shock" $\varepsilon_{\pi,t}$ can be motivated by indexing those prices that are not reoptimized in a given period and by a time-varying elasticity of substitution across goods, leading to time-varying markups.

Equation (69) can be derived from the household consumption Euler equation, where habits in consumption imply that current decisions depend to some extent on last period's decision. The "demand shock" $\varepsilon_{y,t}$ can be attributed to government spending shocks or to movements in the natural level of output.[8]

An empirical version of this model, suitable for quarterly data and similar to that

[7] For the simple example used here, such a state-space representation is given by

$$\begin{bmatrix} \varepsilon_{t+1} \\ g_t \\ \mathrm{E}_t y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_\varepsilon & \rho & 0 \\ -\frac{\sigma_\varepsilon}{\alpha} & -\frac{\rho}{\alpha} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ g_{t-1} \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{\gamma}{\alpha} \end{bmatrix} [u_t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [\zeta_{t+1}],$$

for which the variables in the predetermined block are g_{t-1} and ε_t and the variable in the non-predetermined block is y_t.

[8] See Woodford (2003) for a thorough treatment of the New Keynesian model.

Table 1: Parameter values

Inflation		Output		Monetary policy	
μ_π	0.29	μ_y	0.20	β	0.99
$\alpha_{\pi1}$	0.67	β_{y1}	1.15	λ	0.50
$\alpha_{\pi2}$	-0.14	β_{y2}	-0.27	ν	0.10
$\alpha_{\pi3}$	0.40	β_r	0.09		
$\alpha_{\pi4}$	0.07	σ_y	0.833		
α_y	0.13				
σ_π	1.012				

estimated by Rudebusch (2002a), is given by

$$\pi_t = \mu_\pi \mathrm{E}_{t-1}\bar{\pi}_{t+3} + (1-\mu_\pi)\sum_{j=1}^{4}\alpha_{\pi j}\pi_{t-j} + \alpha_y y_{t-1} + \varepsilon_{\pi,t}, \tag{70}$$

$$y_t = \mu_y \mathrm{E}_{t-1}y_{t+1} + \left(1-\mu_y\right)\sum_{j=1}^{2}\beta_{yj}y_{t-j} - \beta_r\left[i_{t-1} - \mathrm{E}_{t-1}\bar{\pi}_{t+3}\right] + \varepsilon_{y,t}, \tag{71}$$

where $\bar{\pi}_t = 1/4\sum_{j=0}^{3}\pi_{t-j}$ is four-quarter inflation and i_t is the nominal federal funds rate (the policy instrument). We generalize the model slightly to include forward-looking behavior in the output gap equation, as in Rudebusch (2002b).[9] The model's parameter estimates, shown in Table 1, are taken from Rudebusch (2002a) and are obtained using OLS (and survey expectations) on quarterly U.S. data from 1968:Q3 to 1996:Q4, except for the parameter μ_y, which is set to the average estimate in Fuhrer and Rudebusch (2004).

The model's key features are that inflation and the output gap are highly persistent, that monetary policy affects the economy only with a lag, and that expectations are formed using period $t-1$ information. Notice, also, that the weights on expected future inflation and output, while consistent with much of the empirical literature, are small relative to many theory-based specifications.

The central bank's objective function is assumed to be

$$\min_{\{i_t\}} \mathrm{E}_0 \sum_{t=0}^{\infty}\beta^t\left[\pi_t^2 + \lambda y_t^2 + \nu i_t^2\right], \tag{72}$$

where we set $\beta = 0.99$, $\lambda = 0.5$, and $\nu = 0.1$. Thus, the central bank sets monetary policy

[9] Rudebusch (2002b) also includes forward-looking behavior in the real interest rate, replacing $i_{t-1} - \mathrm{E}_{t-1}\bar{\pi}_{t+3}$ in equation (71) with $\mu_r\left[\mathrm{E}_t\bar{\imath}_{t+3} - \mathrm{E}_{t-1}\bar{\pi}_{t+4}\right] + (1-\mu_r)\left[\bar{\imath}_{t-1} - \bar{\pi}_{t-1}\right]$. We instead choose the real interest rate specification Rudebusch (2002a) uses, because the model with expected future interest rates cannot easily be written in state-space form. (It is, however, also straightforward to write that model in structural form.)

to avoid volatility in inflation around its target (normalized to zero) and in the output gap around zero (precluding any discretionary inflation bias). In addition, the central bank desires to limit volatility in the nominal interest rate around target (normalized to zero). The concern for misspecification, ϕ, is chosen so that the detection error probability is 0.1, given a sample of 200 observations.[10]

We first calculate impulse responses to unit-sized[11] innovations to inflation ($\varepsilon_{\pi,t}$) and output ($\varepsilon_{y,t}$) under commitment and discretion using the two solution methods. All impulse responses are shown in Figures 1–8, but for an intuitive understanding of the differences between the two solution methods it is sufficient to consider the model's responses to the inflation shock under commitment, shown in Figures 1–2.

Under the nonrobust policy (RE),[12] a shock to inflation is followed by a prolonged period of high inflation, causing the central bank to tighten monetary policy and to raise the interest rate in order to open up a negative output gap, which will reduce inflation over time. An initial increase in inflation of around one percentage point leads to an increase in the interest rate of 122 basis points, which in turn generates a negative output gap with a maximum effect of minus 0.4 percentage point after four to five quarters. Inflation returns slowly to its initial value and is below one-tenth of the initial shock after eleven quarters.

Using the state-space solution method in Figure 1, the misspecification has no effect in the initial period, as stressed in Section 2. Thus, the central bank does not worry about the evil agent increasing the conditional volatility of the shocks, and the outcomes for inflation and the output gap in the worst-case and approximating equilibria coincide in the initial period. In subsequent periods, however, the evil agent's actions, which make inflation more persistent in the worst-case equilibrium, produce a more aggressive policy response and a larger negative output gap: the interest rate is initially raised 196 basis points, and the effect on the output gap is considerably larger and more persistent. In the approximating equilibrium the more aggressive policy implies that the output gap is larger than under the nonrobust policy, and inflation therefore returns to target faster. Thus, the robust policy is more aggressive than the nonrobust policy, and the central bank fears mainly that inflation is more persistent than is reflected in the reference model.[13]

[10]This implies that $\theta = 54.5$ and 57.5 for the state-space method with commitment and discretion, respectively, and $\phi = 94.5$ and 70.0 for the structural-form method.

[11]Note that this implies that the shocks in the inflation and output equations are equal to the standard deviations of the innovations, as the innovation vectors are scaled by matrices containing the standard deviations.

[12]The responses under the nonrobust policy are the same for the two methods.

[13]The optimal rules for the central bank and the evil agent are reported in Tables C.1 and C.2 in Appendix C.

Table 2: Unconditional variances and value of loss function

	$\text{Var}(\pi_t)$	$\text{Var}(y_t)$	$\text{Var}(i_t)$	Loss
(a) State-space method, commitment				
RE	2.289	2.598	12.922	4.729
Worst	3.282	5.361	30.453	8.633
Approx	2.022	3.444	21.043	5.687
(b) Structural-form method, commitment				
RE	2.289	2.598	12.922	4.729
Worst	3.762	7.057	40.137	10.800
Approx	2.222	4.719	30.137	7.361
(c) State-space method, discretion				
RE	2.793	2.282	11.899	4.931
Worst	4.412	4.735	30.347	9.272
Approx	2.340	2.936	19.131	5.549
(d) Structural-form method, discretion				
RE	2.793	2.282	11.899	4.931
Worst	4.259	5.326	35.916	10.045
Approx	2.432	3.565	26.560	6.664

Giordani and Söderlind (2004) obtain qualitatively similar results using a slightly different model.

Using instead the structural-form solution method in Figure 2, the misspecfication has an effect in the initial period because the evil agent increases the variance of the inflation shock. This effect is relatively small, however: while the initial shock in the state-space method is 101 basis points, in the structural-form method it is 106 basis points instead. By itself this difference between the two methods seems of little importance. Nevertheless, due to the persistence of inflation (and output),[14] this small initial difference has long-lived effects. As a consequence, the central bank needs to increase the interest rate substantially more than for the state-space solution method (the initial increase is now 226 basis points), leading to a larger negative output gap.

Similar differences are obtained when policy is formulated with discretion and in response to output shocks (see Figures 3–8). Although the initial period distortion is small, the total effect is substantially larger and leads to quantitatively important differences

[14]The weights on forward-looking expectations in the model are small; the model includes multiple lags of inflation and output, there are one-period control lags from monetary policy to output and from output to inflation, and expectations are dated at $t-1$. As discussed in Dennis and Söderström (2005), all these features increase the backward-looking nature of the model.

between the two methods. That the differences are important is also apparent in Table 2, which shows the unconditional variances of inflation, output, and the interest rate, along with the value of the loss function, equation (72). Under commitment, inflation, output, and the interest rate in the worst-case equilibrium are 15–30 percent more volatile when using the structural-form solution method and loss is 25 percent higher. Similar numbers apply to the approximating equilibrium. Under discretion the differences are slightly smaller, but remain important.

6 Final remarks

Previous approaches to solving robust control problems have employed state-space methods. These methods rely on the reference model being put into a state-space form, which requires that predetermined variables be explicitly identified and separated from nonpredetermined variables. When the reference model is small or when there are relatively few state variables, obtaining a state-space form can be reasonably straightforward. However, as the reference model's complexity increases, manipulating it into state-space form can become a torturously difficult, time-consuming, and error-prone task. For nonrobust control problems, difficulties with obtaining a state-space representation can generally be overcome by using the structural-form solution methods developed by Dennis (2005a).

In this paper we show how Dennis's (2005a) structural-form solution methods can be applied to robust control problems, thereby making it easier to analyze complex models using robust control methods. As an additional contribution, we show that, upon departing from rational expectations, the structural-form methods need not generate the same equilibrium behavior as the state-space methods. In particular, whereas the state-space methods, as they are typically applied, result in misspecifications that distort the conditional means of the shock processes, for the structural-form methods the misspecifications distort both the conditional means and the conditional variance/covariances of the shocks. We show that different misspecifications emerge in equilibrium because the two solution methods are solving different, but closely related, problems. In particular, differences arise because the state-space methods, by forcing shocks to serve as states when they are not predetermined, restrict the state vector in ways that the structural form solution methods do not. When these restrictions are either relaxed in the state-space representation or imposed on the structural-form representation, the two approaches return identical solutions. To accommodate the distortions to the conditional volatility of the shocks, we generalize the existing method for calculating detection-error probabilities.

We illustrate the structural-form solution methods by applying them to an empirical New Keynesian business cycle model of the genre widely used to study monetary policy under rational expectations. A key finding from this exercise is that the strategically designed specification errors will tend to distort the Phillips curve in an effort to make inflation more persistent, and hence harder and more costly to stabilize. The optimal response to these distortions is for the central bank to become more activist in its response to shocks. Finally, with the New Keynesian model serving as a laboratory, we show that, separate to whether policy is set with commitment or discretion, the distortions to the conditional volatility of the shocks that the structural-form methods generate have implications for monetary policy and for economic outcomes that are both qualitatively and quantitatively important.

A The discretionary equilibrium using the state-space method

In the discretionary case the optimization problem remains

$$\min_{\{\mathbf{u}_t\}} \max_{\{\mathbf{v}_{t+1}\}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\mathbf{z}_t' \mathbf{W} \mathbf{z}_t + 2\mathbf{z}_t' \widetilde{\mathbf{U}} \widetilde{\mathbf{u}}_t + \widetilde{\mathbf{u}}_t' \widetilde{\mathbf{R}} \widetilde{\mathbf{u}}_t \right], \tag{A1}$$

subject to

$$\mathbf{z}_{1t+1} = \mathbf{A}_{11}\mathbf{z}_{1t} + \mathbf{A}_{12}\mathbf{z}_{2t} + \widetilde{\mathbf{B}}_1 \widetilde{\mathbf{u}}_t + \mathbf{C}_1 \boldsymbol{\varepsilon}_{1t+1}, \tag{A2}$$

$$\mathrm{E}_t \mathbf{z}_{2t+1} = \mathbf{A}_{21}\mathbf{z}_{1t} + \mathbf{A}_{22}\mathbf{z}_{2t} + \widetilde{\mathbf{B}}_2 \widetilde{\mathbf{u}}_t, \tag{A3}$$

but now neither the policymaker nor the evil agent can commit. A convenient way to solve this dynamic optimization problem is to apply the method presented in Backus and Driffill (1986). Conjecturing that the solution for the non-predetermined variables in period $t+1$ has the form

$$\mathbf{z}_{2t+1} = \mathbf{H}\mathbf{z}_{1t+1}, \tag{A4}$$

equations (A2)–(A4) imply that the non-predetermined variables, $\mathbf{z}_{2t}$, depend on the predetermined variables, $\mathbf{z}_{1t}$, and the control variables, $\widetilde{\mathbf{u}}_t$, according to

$$\mathbf{z}_{2t} = \mathbf{J}\mathbf{z}_{1t} + \mathbf{K}\widetilde{\mathbf{u}}_t, \tag{A5}$$

where

$$\mathbf{J} \equiv \left(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1} \left(\mathbf{A}_{21} - \mathbf{H}\mathbf{A}_{11}\right), \tag{A6}$$

$$\mathbf{K} \equiv \left(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1} \left(\widetilde{\mathbf{B}}_2 - \mathbf{H}\widetilde{\mathbf{B}}_1\right). \tag{A7}$$

Using (A5) to substitute the non-predetermined variables out of the objective function, the dynamic program for the optimization problem with discretion is

$$\mathbf{z}_{1t}'\mathbf{P}\mathbf{z}_{1t} + k \equiv \min_{\mathbf{u}_t} \max_{\mathbf{v}_{t+1}} [\mathbf{z}_{1t}' \overline{\mathbf{W}} \mathbf{z}_{1t} + 2\mathbf{z}_{1t}' \overline{\mathbf{U}} \widetilde{\mathbf{u}}_t + \widetilde{\mathbf{u}}_t' \overline{\mathbf{R}} \widetilde{\mathbf{u}}_t + \beta \mathrm{E}_t \left(\mathbf{z}_{1t+1}' \mathbf{P} \mathbf{z}_{1t+1} + k\right)], \tag{A8}$$

where

$$\overline{\mathbf{W}} \equiv \mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{J} + \mathbf{J}'\mathbf{W}_{21} + \mathbf{J}'\mathbf{W}_{22}\mathbf{J}, \tag{A9}$$

$$\overline{\mathbf{U}} \equiv \mathbf{W}_{12}\mathbf{K} + \mathbf{J}'\mathbf{W}_{22}\mathbf{K} + \widetilde{\mathbf{U}}_1 + \mathbf{J}'\widetilde{\mathbf{U}}_2', \tag{A10}$$

$$\overline{\mathbf{R}} \equiv \mathbf{K}'\mathbf{W}_{22}\mathbf{K} + \widetilde{\mathbf{U}}_2'\mathbf{K} + \mathbf{K}'\widetilde{\mathbf{U}}_2 + \widetilde{\mathbf{R}}, \tag{A11}$$

and its solution is given by

$$\begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_{t+1} \end{bmatrix} = -\mathbf{F}\mathbf{z}_{1t}, \tag{A12}$$

$$\mathbf{z}_{2t} = (\mathbf{J} - \mathbf{K}\mathbf{F})\,\mathbf{z}_{1t}, \tag{A13}$$

$$\mathbf{z}_{1t+1} = \left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} - \widetilde{\mathbf{B}}_1\mathbf{F}\right)\mathbf{z}_{1t} + \mathbf{C}_1\boldsymbol{\varepsilon}_{1t+1}, \tag{A14}$$

where $\mathbf{P}$ and $\mathbf{F}$ are obtained by solving for the fix-point of

$$\mathbf{J} \equiv (\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})^{-1}(\mathbf{A}_{21} - \mathbf{H}\mathbf{A}_{11}), \tag{A15}$$

$$\mathbf{K} \equiv (\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})^{-1}\left(\widetilde{\mathbf{B}}_2 - \mathbf{H}\widetilde{\mathbf{B}}_1\right), \tag{A16}$$

$$\widetilde{\mathbf{A}}_{11} \equiv \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}, \tag{A17}$$

$$\widetilde{\mathbf{A}}_{12} \equiv \mathbf{A}_{12}\mathbf{K} + \widetilde{\mathbf{B}}_1, \tag{A18}$$

$$\mathbf{P} = \overline{\mathbf{W}} - 2\overline{\mathbf{U}}\mathbf{F} + \mathbf{F}'\overline{\mathbf{R}}\mathbf{F} + \beta\left(\widetilde{\mathbf{A}}_{11} - \widetilde{\mathbf{A}}_{12}\mathbf{F}\right)'\mathbf{P}\left(\widetilde{\mathbf{A}}_{11} - \widetilde{\mathbf{A}}_{12}\mathbf{F}\right), \tag{A19}$$

$$\mathbf{F} = \left(\overline{\mathbf{R}} + \beta\widetilde{\mathbf{A}}_{12}'\mathbf{P}\widetilde{\mathbf{A}}_{12}\right)^{-1}\left(\overline{\mathbf{U}}' + \beta\widetilde{\mathbf{A}}_{12}'\mathbf{P}\widetilde{\mathbf{A}}_{11}\right), \tag{A20}$$

$$\mathbf{H} = (\mathbf{J} - \mathbf{K}\mathbf{F}). \tag{A21}$$

With the worst-case equilibrium given by equations (A12)–(A14), partitioning $\mathbf{F}$ into $[\ \mathbf{F}_{\mathbf{u}}' \ \ \mathbf{F}_{\mathbf{v}}'\ ]'$ where $\mathbf{F}_{\mathbf{u}}$ and $\mathbf{F}_{\mathbf{v}}$ are conformable with $\mathbf{u}_t$ and $\mathbf{v}_{t+1}$, respectively, the approximating equilibrium is derived from equations (A12)–(A14) by setting $\mathbf{F}_{\mathbf{v}} = \mathbf{0}$. For further details see Giordani and Söderlind (2004).

B Setting up the model

To write the Rudebusch (2002a) model in state-space form, first lead (70) and (71) one period:

$$\pi_{t+1} = \frac{\mu_\pi}{4}\mathrm{E}_t\left[\pi_{t+1}+\pi_{t+2}+\pi_{t+3}+\pi_{t+4}\right] + (1-\mu_\pi)\left[\alpha_{\pi 1}\pi_t+\alpha_{\pi 2}\pi_{t-1}+\alpha_{\pi 3}\pi_{t-2}+\alpha_{\pi 4}\pi_{t-3}\right]+\alpha_y y_t+\varepsilon_{\pi,t+1}, \tag{B1}$$

$$y_{t+1} = \mu_y \mathrm{E}_t y_{t+2} + \left(1-\mu_y\right)\left[\beta_{y1}y_t+\beta_{y2}y_{t-1}\right] -\beta_r\left[i_t-\frac{1}{4}\mathrm{E}_t\left(\pi_{t+1}+\pi_{t+2}+\pi_{t+3}+\pi_{t+4}\right)\right]+\varepsilon_{y,t+1}. \tag{B2}$$

Then solve for the forward-looking variables $\mathrm{E}_t\pi_{t+4}$ and $\mathrm{E}_t y_{t+2}$ and take expectations as of period t:

$$\frac{\mu_\pi}{4}\mathrm{E}_t\pi_{t+4} = \left(1-\frac{\mu_\pi}{4}\right)\mathrm{E}_t\pi_{t+1}-\frac{\mu_\pi}{4}\mathrm{E}_t\pi_{t+2}-\frac{\mu_\pi}{4}\mathrm{E}_t\pi_{t+3} -(1-\mu_\pi)\left[\alpha_{\pi 1}\pi_t+\alpha_{\pi 2}\pi_{t-1}+\alpha_{\pi 3}\pi_{t-2}+\alpha_{\pi 4}\pi_{t-3}\right]-\alpha_y y_t, \tag{B3}$$

$$\mu_y\mathrm{E}_t y_{t+2} + \frac{\beta_r}{4}\mathrm{E}_t\pi_{t+4} = \mathrm{E}_t y_{t+1}-\left(1-\mu_y\right)\left[\beta_{y1}y_t+\beta_{y2}y_{t-1}\right] +\beta_r\left[i_t-\frac{1}{4}\mathrm{E}_t\left(\pi_{t+1}+\pi_{t+2}+\pi_{t+3}\right)\right], \tag{B4}$$

and reintroduce the disturbances via

$$\pi_{t+1} = \mathrm{E}_t\pi_{t+1}+\varepsilon_{\pi,t+1}, \tag{B5}$$

$$y_{t+1} = \mathrm{E}_t y_{t+1}+\varepsilon_{y,t+1}. \tag{B6}$$

Define an $(n_1\times 1)$ vector $(n_1=6)$ of predetermined state variables as

$$\mathbf{z}_{1t} = \left\{\pi_t,\pi_{t-1},\pi_{t-2},\pi_{t-3},y_t,y_{t-1}\right\}', \tag{B7}$$

an $(n_2\times 1)$ vector $(n_2=4)$ of non-predetermined variables as

$$\mathbf{z}_{2t} = \left\{\mathrm{E}_t\pi_{t+1},\mathrm{E}_t\pi_{t+2},\mathrm{E}_t\pi_{t+3},\mathrm{E}_t y_{t+1}\right\}', \tag{B8}$$

and an $(s\times 1)$ vector $(s=2)$ of innovations as

$$\boldsymbol{\varepsilon}_{1t} = \left\{\varepsilon_{\pi t},\varepsilon_{yt}\right\}'. \tag{B9}$$

Also define the policy instrument as $\mathbf{u}_t=\{i_t\}$. We can then write the model in compact form as

$$\mathbf{A}_0 \begin{bmatrix} \mathbf{z}_{1t+1} \\ \mathrm{E}_t \mathbf{z}_{2t+1} \end{bmatrix} = \mathbf{A}_1 \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} + \mathbf{B}_1 \mathbf{u}_t + \mathbf{C}_1 \boldsymbol{\varepsilon}_{1t+1}. \tag{B10}$$

Assuming that $\mathbf{A}_0$ is nonsingular, the usual state-space form can be obtained by premultiplying (B10) by $\mathbf{A}_0^{-1}$ to get

$$\begin{bmatrix} \mathbf{z}_{1t+1} \\ \mathrm{E}_t \mathbf{z}_{2t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} + \mathbf{B} \mathbf{u}_t + \mathbf{C}_1 \boldsymbol{\varepsilon}_{1t+1}, \tag{B11}$$

where $\mathbf{A} = \mathbf{A}_0^{-1}\mathbf{A}_1$ and $\mathbf{B} = \mathbf{A}_0^{-1}\mathbf{B}_1$.[15]

Writing the model in structural form is more straightforward, as it does not require any rearrangement of the equations. Define the $(n \times 1)$ vector $(n = 13)$ of endogenous variables as

$$\mathbf{y}_t = \{\mathrm{E}_t\pi_{t+4}, \mathrm{E}_t\pi_{t+3}, \mathrm{E}_t\pi_{t+2}, \mathrm{E}_t\pi_{t+1}, \pi_t, \pi_{t-1}, \pi_{t-2}, \pi_{t-3}, \mathrm{E}_t y_{t+2}, \mathrm{E}_t y_{t+1}, y_t, y_{t-1}, i_t\}', \tag{B12}$$

an $(s \times 1)$ vector $(s = 2)$ of innovations as

$$\boldsymbol{\varepsilon}_t = \{\varepsilon_{\pi t}, \varepsilon_{yt}\}', \tag{B13}$$

and define the policy instrument as $\mathbf{u}_t = \{i_t\}$. Then it is straightforward to write the model on the required form

$$\mathbf{A}_0 \mathbf{y}_t = \mathbf{A}_1 \mathbf{y}_{t-1} + \mathbf{A}_2 \mathrm{E}_t \mathbf{y}_{t+1} + \mathbf{A}_3 \mathbf{u}_t + \mathbf{A}_4 \boldsymbol{\varepsilon}_t. \tag{B14}$$

[15] Note that $\mathbf{A}_0^{-1}\mathbf{C}_1 = \mathbf{C}_1$, since $\mathbf{A_0}$ is block diagonal with an identity matrix as its upper left block and the lower block of $\mathbf{C}_1$ is zero.

C Optimal policy rules and misspecification

Table C.1: Optimal policy rules and misspecification, state-space method

	Coefficient on					
	π_t	π_{t-1}	π_{t-2}	π_{t-3}	y_t	y_{t-1}
(a) Policy rules, commitment						
RE	1.202	0.470	0.526	0.076	2.000	−0.547
Worst	1.940	0.744	0.847	0.123	2.557	−0.685
(b) Policy rules, discretion						
RE	1.330	0.518	0.582	0.084	2.129	−0.582
Worst	2.137	0.817	0.932	0.135	2.745	−0.736
(c) Misspecification, commitment						
v_π	0.071	0.023	0.034	0.005	0.045	−0.010
v_y	0.033	0.013	0.014	0.002	0.043	−0.012
(d) Misspecification, discretion						
v_π	0.071	0.023	0.033	0.005	0.046	−0.010
v_y	0.034	0.013	0.015	0.002	0.044	−0.012

Table C.2: Optimal policy rules and misspecification, structural-form method

	Coefficient on										
	$\mathrm{E}_{t-1}\pi_{t\ldots t+3}$	π_{t-1}	π_{t-2}	π_{t-3}	π_{t-4}	$\mathrm{E}_{t-1}y_{t+1}$	y_{t-1}	y_{t-2}	i_{t-1}	$\varepsilon_{\pi t}$	ε_{yt}
(a) Policy rules, commitment											
RE	0.132	1.042	0.407	0.417	0.060	0.400	1.449	−0.432	−0.180	1.216	1.666
Worst	0.224	1.906	0.743	0.776	0.111	0.552	2.097	−0.596	−0.248	2.265	2.297
(b) Policy rules, discretion											
RE	0.144	1.150	0.449	0.462	0.066	0.426	1.549	−0.460	−0.192	1.346	1.774
Worst	0.264	2.267	0.883	0.926	0.133	0.626	2.403	−0.677	−0.282	2.705	2.609
(c) Misspecification, commitment											
v_π	0.004	0.036	0.012	0.017	0.002	0.005	0.024	−0.005	−0.002	0.050	0.021
v_y	0.002	0.017	0.007	0.007	0.001	0.005	0.022	−0.006	−0.002	0.021	0.023
(d) Misspecification, discretion											
v_π	0.006	0.055	0.019	0.025	0.004	0.008	0.037	−0.009	−0.004	0.075	0.033
v_y	0.003	0.027	0.011	0.011	0.002	0.008	0.033	−0.009	−0.004	0.033	0.034

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Figure 1: Response to inflation shock, State-space method with commitment

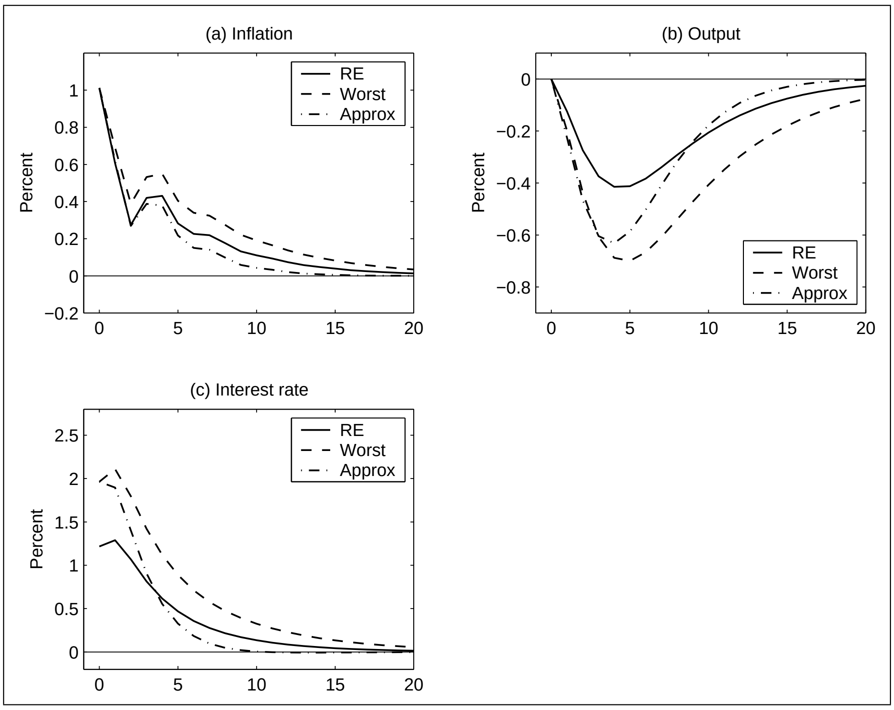

Figure 2: Response to inflation shock, Structural-form method with commitment

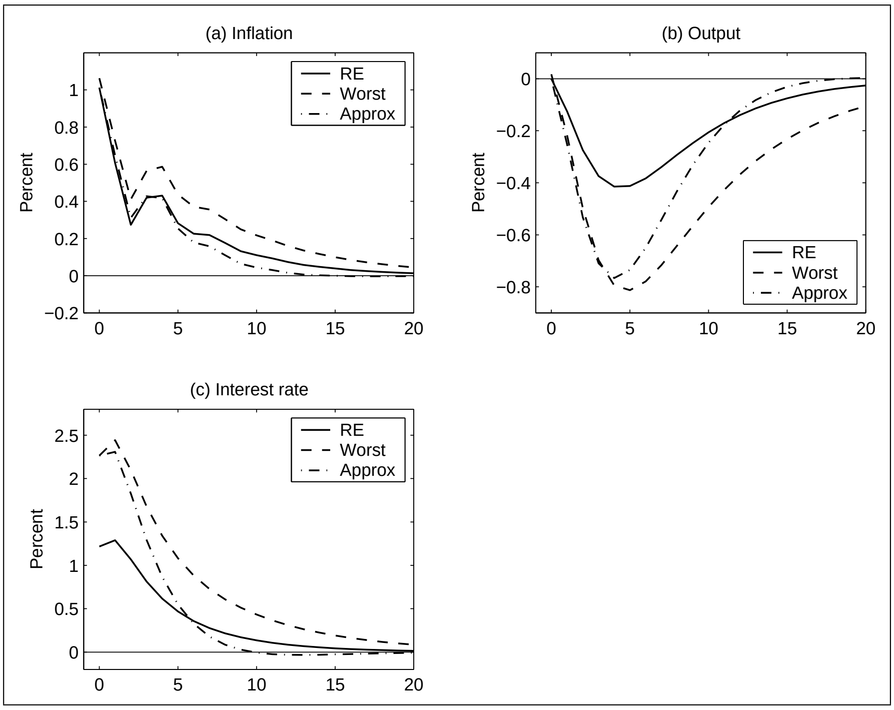

Figure 3: Response to output shock, State-space method with commitment

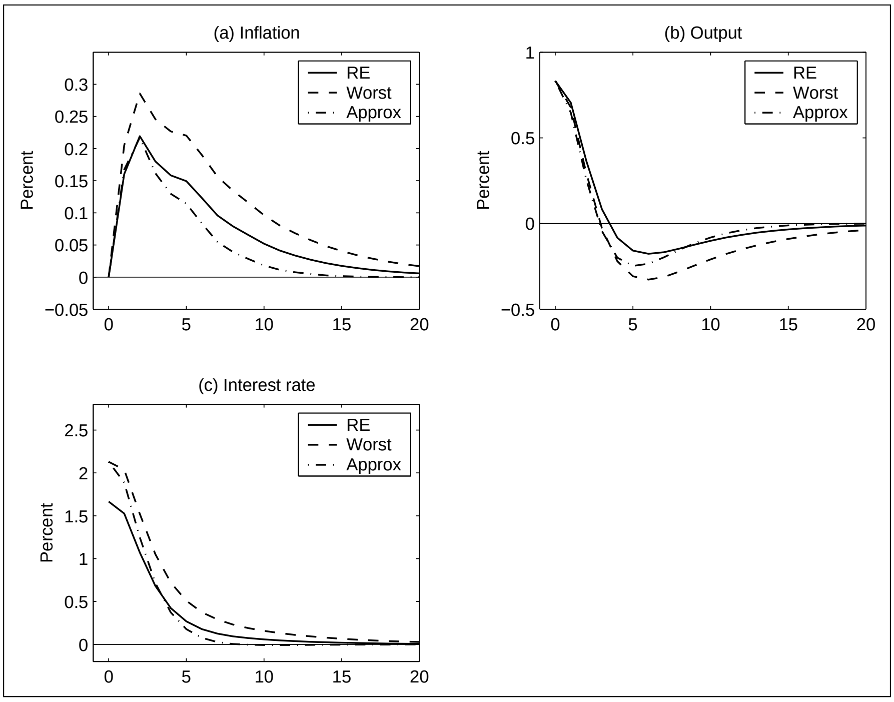

Figure 4: Response to output shock, Structural-form method with commitment

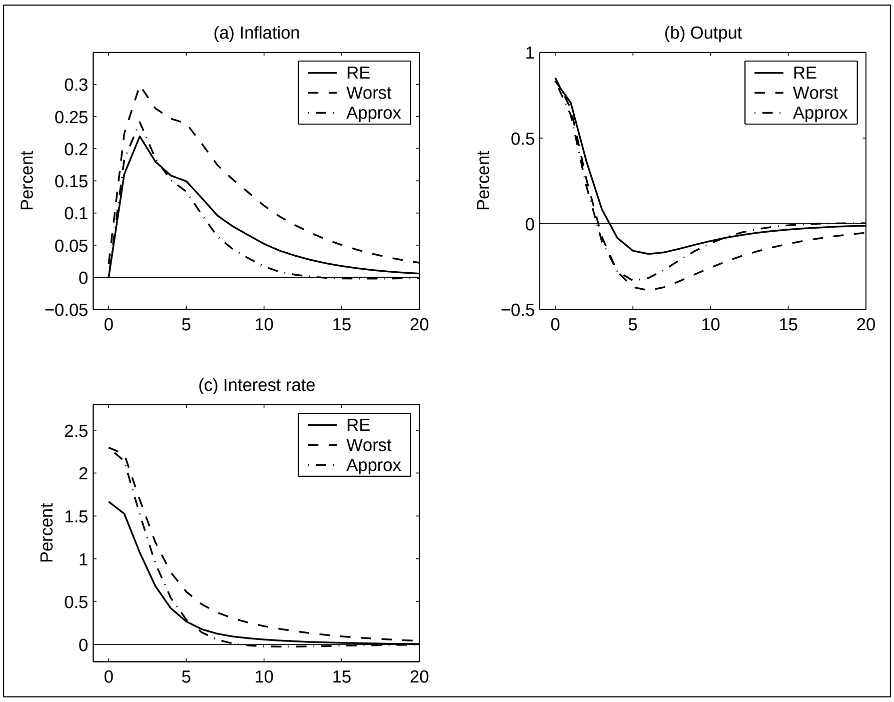

Figure 5: Response to inflation shock, State-space method with discretion

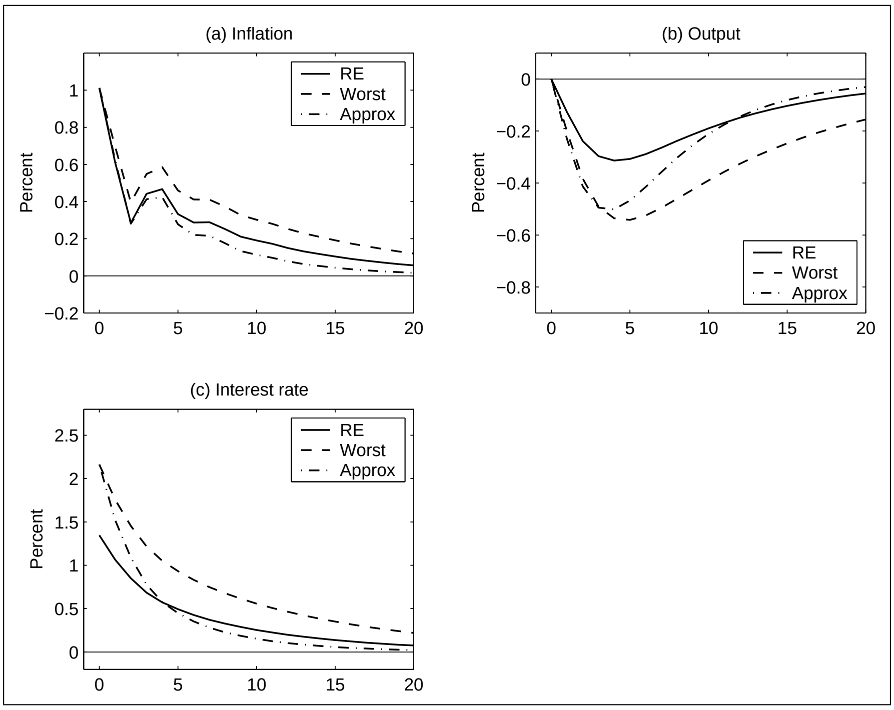

Figure 6: Response to inflation shock, Structural-form method with discretion

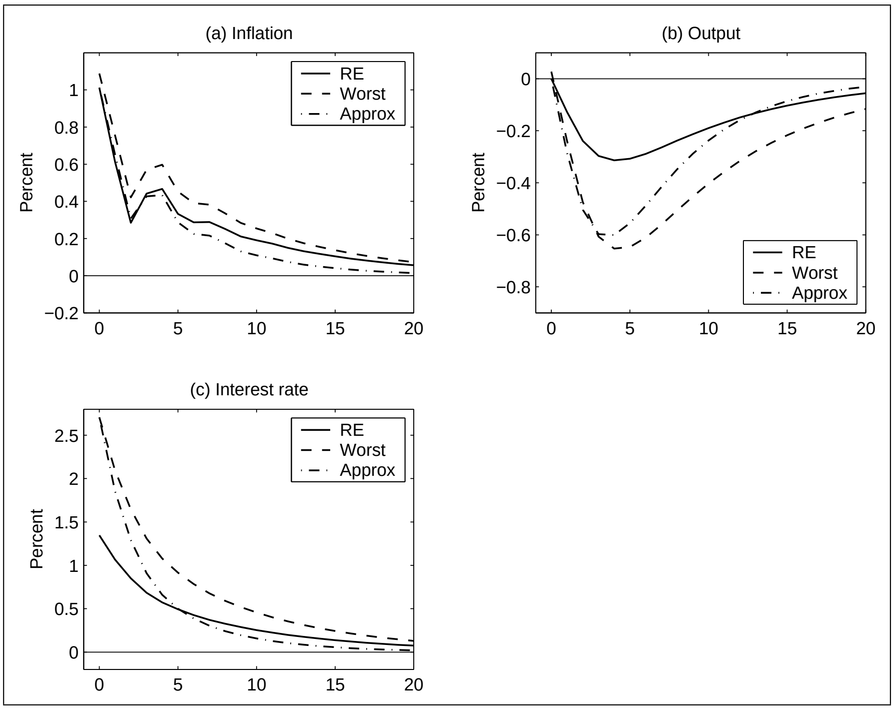

Figure 7: Response to output shock, State-space method with discretion

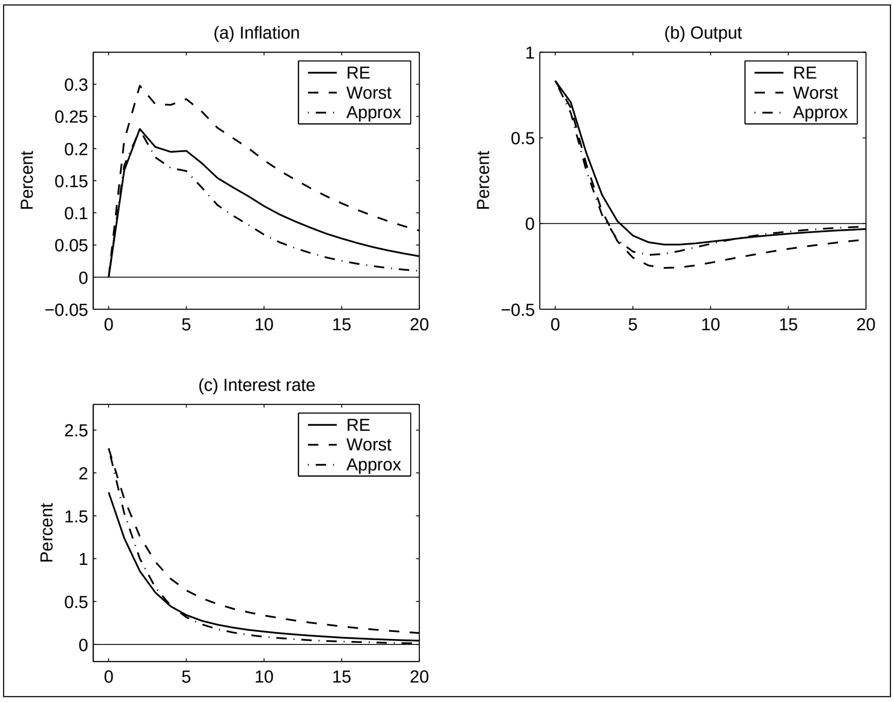

Figure 8: Response to output shock, Structural-form method with discretion