Dynamic Suboptimality of Competitive Equilibrium in Multiperiod Overlapping Generations Economies

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September 21, 2005

Abstract

The question we ask is: within the set of a three-period-lived OLG economies with a stochastic endowment process, a stochastic dividend process, and sequentially complete markets, under what set of conditions may a set of government transfers dynamically Pareto dominate the *laissez faire* equilibrium? We start by characterizing perfect risk sharing and find that it implies a strongly stationary set of state-dependent consumption claims. We also derive the stochastic equivalent of the deterministic steady-state by steady-state optimal marginal rate of substitution. We show then that the risk sharing of the recursive competitive *laissez faire* equilibrium of any overlapping generations economy with weakly more than three generations is non-stationary and that risk is suboptimally shared. We then show that we can construct a sequence of consumption allocations that only depends on the exogenous state and which Pareto dominate the *laissez faire* allocations in an *ex interim* as well as *ex ante* sense. We also redefine *conditional* Pareto optimality to apply within this framework and show that under a broad set of conditions, there also exists a sequence of allocations that dominates the *laissez faire* equilibrium in this sense. Finally, we apply these tools and results to an economy where the endowment is constant, but where fertility is stochastic, i.e. the number of newborn individuals who enters the economy follows a Markov Process.

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1 Introduction

Within the set of a three-period lives OLG economies with a stochastic endowment process, under what broad set of conditions may government transfers dynamically Pareto dominate a complete-market Lucas-tree economy?

Theoretically, a fundamental result about OLG models is that within the class of n-period-lived OLG economies with a deterministic endowment process, the "classical case" is dynamically Pareto efficient, but not Pareto optimal in steady-state by steady-state comparison. In stochastic extensions of this class of models, prices satisfy a version of the Cass criterion, and hence allocations are optimal as long as risk-sharing among agents is also optimal. We will show here that the *laissez-faire* competitive equilibrium for these models does not allocate risk optimally, and that it is possible to Pareto dominate this equilibrium. To establish this result, we will first show generally that optimal risk-sharing requires that allocations be strongly stationary, in the sense that the time t allocation depends only on the realization of the exogenous shock at time t. Since the *laissez-faire* competitive equilibrium necessarily depends on lagged endogenous variables, it is not strongly stationary and hence allocates risk suboptimally.

We then show that the stochastic extension of the Pareto optimal steady-state known to exist in the deterministic model is, in fact, strongly stationary, and hence Pareto optimal for the stochastic model. We derive a set of government taxes and transfers which will implement this stochastic steady-state in the model, and show, via the second welfare theorem, that the steady-state can also be supported as a competitive equilibrium after imposition of the required lump sum taxes and transfers.

The nature of the Pareto improvement as we move from the *laissez-faire* equilibrium to the optimal equilibrium can be studied using simulations. Simulations are required because the *laissez-faire* competitive equilibrium must be generated recursively. In the simulations, it is apparent that old agents bear too much risk, since their income is derived entirely from savings and investment in the productive asset. As dividends and prices fluctuate, so does retirement income. Young and middle-aged cohorts face too little risk, save too much and consume too little. At the optimal allocation, old agents accept reductions in average consumption in exchange for reductions in the variance of their allocations. Middle-aged and young (in an *ex ante* sense) trade increased risk for increases in average consumption levels.

The specific welfare comparisons we make are drawn from Demange (2002) optimality criteria. In particular, we consider conditional or in-

terim, optimality, in which young agents born into different states of nature are treated as if they were different individuals. We also consider the less constrained notion of *ex ante* optimality in which the young are treated as if they were able to make choices prior to their birth. This optimality criterion is the appropriate one for a benevolent and long-lived central planner. In the context of specific Pareto optimality criteria, we also use the notion of short-run optimality, which limits consideration of Pareto improvements to finite numbers of agents, to establish our main result on the inoptimality of the competitive equilibrium.

For applied work, these results have clear implications for how one should approach questions regarding funded vs. non-funded social pension systems and all models regarding intertemporal and intergenerational risk-sharing and savings behavior over the life cycle. The results we obtain also bear on the current debate over Social Security privatization. When one examines the overall set of intertemporal government transfers in the U.S. economy, it becomes apparent that the U.S. economy is not at a *laissez-faire* equilibrium. Rather, we are at one in which there are active government taxes and transfers at work. The most obvious of these is the Social Security system. But this isn't the only transfer at work. There are additional transfers in place which actually benefit young and middle-aged households. By an application of the benefit principle for taxation, the younger families who receive these benefits should also pay for them, though at a later stage in life when they are better able to afford it. From this perspective, these benefits amount to a transfer from older households to younger ones, in compensation for the benefits received earlier by the older households.

From these observations, we can view reforms to eliminate these intergenerational income transfers as an attempt to move the U.S. economy back toward something approximating the *laissez-faire* equilibrium. Our results suggest that this might not be a good idea.

1.1 Literature

Economists have been concerned with the optimality of competitive equilibrium allocations in a dynamic, stochastic economies since the first discovery that dynamic economies can exhibit phenomena that lead to competitive allocations which are not Pareto optimal. The Cass criterion provides a way for determining whether an allocation is suboptimal – generally due to capital overaccumulation – in the context of the neo-classical growth model using the competitive equilibrium prices associated with the allocation (Cass, 1972). Gale (1973) invented the following terminology: when

$mrs < 1$ he calls this the "Samuelsonian case", whereas when $mrs > 1$ he calls this the "classical case". He further established that the "Samuelsonian case" is dynamically Pareto suboptimal, whereas it is not possible to dynamically Pareto dominate the "classical case".

In the context of overlapping generations models, most analyses of the welfare properties of equilibrium have been undertaken in the context of the simplest version of the model in which agents live two periods, and trade a single good, generally called "consumption". The earliest examination of welfare properties in stochastic models were by Muench (1977), Peled (1982), and Aiyagari and Peled (1991). In the simple model, the competitive equilibrium prices and allocations are always strongly stationary, in the sense that endogenously determined variables are functions of the exogenous shocks alone. Hence, for models in which the exogenous shock is taken to have finite support, the model can be analyzed using standard finite-dimensional vector space techniques. The Peled and Aiyagari and Peled papers exploit this fact to show that if the competitive equilibrium is Pareto optimal, then the dominant root of the pricing kernel (i.e. the matrix of state contingent asset prices) at the equilibrium allocation will be strictly less than one.

These results have been extended in a number of directions. Work by Abel, Mankiw, Summers and Zeckhauser (1989) characterized the efficiency properties of competitive equilibrium in the benchmark model with production using the Cass criterion. Zilcha (1990) provides a similar characterization. Chattopadhyay and Gottardi (1999) show the optimality of competitive equilibrium in a two-period-lived agents model with more than one good traded in each period. This extension is not trivial, since Spear (1985) shows that strongly stationary equilibrium don't exist generically when agents trade more than a single good in each period. A similar result obtains for single commodity models if agents live more than two periods. Finally, Demange (2002) provides a general characterization of various notions of optimality in stochastic OLG settings, and shows that the stationary competitive equilibrium in a model in which agents trade a single good and live more than two periods satisfies the Cass criterion, and hence will be optimal except for possible inefficiencies in risk-sharing. Our work extends Demange's in showing that not only does the competitive equilibrium for this model not allocate risk efficiently, but it is also feasible to construct a sequence of state contingent consumptions to a strongly stationary equilibrium.

The observation that the risk sharing among the individuals in the society might be better with a social security system than without, has been

made by among others Ball and Mankiw (2001), Bohn (2001), and Smetters (2004). All these contributions fail, however, in discussing the dynamic Pareto properties of the equilibria.

2 The Model

2.1 The Deterministic Benchmark Economy

We work with an overlapping generations model in which agents become economically active at age 20, and live for three 20 year periods, which we call youth, middle-age, and retirement. Households in the model receive a deterministic labor income when young (w_y) and middle-aged (w^m). They have two assets available: bonds which are in zero net supply and pay one unit of consumption next period, and equity, which are in fixed supply, normalized to one. Each period a dividend δ is paid out to the equity holders.

Agent's preferences are given by a utility function

$$U\left(c_y, c_m, c_r\right) = u\left(c_y\right) + \beta u\left(c_m\right) + \beta^2 u\left(c_r\right)$$

where the discount factor β is such that $0 < \beta \leq 1$. The period utility functions $u\left(\cdot\right)$ are strictly concave, strictly increasing and satisfy Inada conditions. For the deterministic market economy, agents demands are solutions to the optimization problem

$$\max_{\left(c_y, c_m, c_r\right)} u\left(c_y\right) + \beta u\left(c_m\right) + \beta^2 u\left(c_r\right)$$

$$c_y = w_y - qb_y - pe_y$$
$$c_m = w_m + b_y + \left(p' + \delta\right) e_y - q'b_m - p'e_m$$
$$c_r = b_m + \left(p'' + \delta\right) e_m$$

where q, q' and q'' are bond prices in each of the agent's three periods of life, and p, p' and p'' are asset prices.

The market clearing conditions for the model are

$$b_y + b_m = 0,$$
$$e_y + e_m = 1,$$

and the overall resource constraint

$$c_y + c_m + c_r \leq w_y + w_m + \delta = \omega.$$

In the context of this deterministic model, we show first that the solution to the central planners optimization problem leads to an allocation in which the interest rate is zero, and which is not the competitive equilibrium allocation.

Proposition 1. *For the deterministic benchmark model with $\delta = 0$, the optimal steady-state allocation has the interest rate equal to one.*

Proof. The social planner's problem for this case is

$$c_y + c_m + c_r = w_y + w_m$$

The first-order conditions for this problem are

$$
\begin{aligned}
u'(c_y) - \lambda &= 0 \\
\beta u'(c_m) - \lambda &= 0 \\
\beta^2 u'(c_r) - \lambda &= 0.
\end{aligned}
$$

Now, consider the competitive steady-state equilibrium. For this version of the model, each agent solves

$$\max_{(c_y, c_m, c_r)} u(c_y) + \beta u(c_m) + \beta^2 u(c_r)$$

$$c_y = w_y - q b_y$$
$$c_m = w^m + b_y - q' b_m$$
$$c_r = b_m$$

The first-order conditions for this problem are

$$
\begin{aligned}
u'(c_y) q &= \beta u'(c_m) \\
u'(c_m) q' &= \beta u'(c_r).
\end{aligned}
$$

From the first-order conditions, it is clear that the steady-state with $q = q' = 1$ generates the optimal allocation. One can show that any equilibrium sequence with $q_t < 1$ for all t generates a sequence of allocations which eventually becomes infeasible. $\square$

Now consider the benchmark economy with a productive asset which yields a dividend of $\delta > 0$ in each period, and which is owned by the initial old generation. The social optimum for this version of the model is the same as in the first benchmark, except that total resources in every period have increased by δ. The marginal rates of substitution at the optimal allocation

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are the same as in the first case, however, so that the implicit interest rate at the social optimum is zero. The competitive equilibrium for this version is considerably different, though. Agents optimizations for this case are

$$\max_{(c_y, c_m, c_r)} u(c_y) + \beta u(c_m) + \beta^2 u(c_r)$$

$$c_y = w_y - q b_y - p e_y$$
$$c_m = w_m + b_y + (p' + \delta) e_y - q' b_m - p' e_m$$
$$c_r = b_m + (p'' + \delta) e_m$$

where e^i is the type $i = y, m$ agent's holdings of the asset and the p's are the asset prices. The first-order conditions for this problem are

$$u'(c_y) q = \beta u'(c_m)$$
$$u'(c_y) p = \beta u'(c_m)(p' + \delta)$$
$$u'(c_m) q' = \beta u'(c_r)$$
$$u'(c_m) p' = \beta u'(c_r)(p'' + \delta).$$

Market-clearing requires that $b^y + b^m = 0$ and $e^y + e^m = 1$. At a steady-state equilibrium, the first-order conditions imply that

$$p = q(p + \delta)$$

or

$$p = \frac{\delta}{1 - q}.$$

Since the asset is productive, it will have positive value and hence, at the steady-state competitive equilibrium, $q < 1$. Since

$$q = \frac{p}{p + \delta}$$

it also follows that as δ gets larger, q becomes monotonically smaller. It is also clear that when $\delta = 0$, $q = 1$, and hence, that as δ gets larger, the competitive equilibrium allocation diverges from the socially optimal steady-state.

While the competitive equilibrium for the benchmark economy with a productive asset is not Pareto optimal in a steady-state-by-steady-state comparison, the CE for this benchmark case is *dynamically* Pareto optimal, since any reallocation away from the CE steady-state toward the socially optimal allocation necessarily makes old agents worse off.

For later reference, we note that since the social optimum for the benchmark model constitutes a steady-state Pareto optimum, the second welfare theorem implies that this allocation can be obtained as a competitive equilibrium after imposing lump sum transfers which allocate the dividend from the asset to the three types of agents. We can also implement this via a proportional tax on the dividend. For this variation on the model, the budget constraints take the form

$$c_y = w_y - qb_y - pe_y + \tau_y$$
$$c_m = w_m + b_y + [p + (1 - t)\,\delta]\,e_y - qb_m - pe_m + \tau_m$$
$$c_r = b_m + [p + (1 - t)\,\delta]\,e_m$$

where we have imposed that prices be at their steady-state equilibrium values, t is the tax rate on the return to the asset, and τ^i, $i = y, m, r$ is a lump sum transfer that each agent receives from the government. The government budget constraint is

$$t\delta - \tau = 0$$

where $\tau = \tau^y + \tau^m + \tau^r$. For this version of the benchmark model, the bond and asset prices are related (via the first-order conditions) by

$$p = (1 - t)\,\frac{q\delta}{1 - q}.$$

Note next that when $t = 1$, $p = 0$ and $\tau = \delta$, so that the asset has effectively been nationalized. In this case, the steady-state equilibrium allocation will be the socially optimal steady-state. By continuity, there will be tax rates $t < 1$ at which the resulting competitive steady-state dominates (in steady-state comparisons) the competitive equilibrium at $t = 0$. We emphasize, however, that if we start at the $t = 0$ steady-state and then impose a positive tax on the asset return, we make the old generation at the time of the tax regime change worse off.

2.2 The Stochastic Economy

As in the deterministic benchmark case, we denote total endowment by ω. For the stochastic extension of the model the exogenous state of the economy in period t is given by $s_t \in S_t$, $t = 1, 2, 3, \ldots$, where $S = \{1, 2, 3, \ldots, S\}$ (and S both denotes the set of exogenous states and the number of exogenous

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states). The exogenous state s_t follows a Markov process with time-invariant transition probabilities given by

$$\pi^{s,s'} = \Pr\left\{s_t = s' \mid s_{t-1} = s\right\}.$$

For simplicity of notation, let us here assume that the number of exogenous states in S_t is 2: $S_t = \{l, h\}$. We can then write the time-invariant transition matrix

$$\Pi = \begin{bmatrix} 1 - \pi^{h,l} & \pi^{h,l} \\ \pi^{l,h} & 1 - \pi^{l,h} \end{bmatrix}.$$

Unconditional expected total per period aggregate endowment is normalized to 1. The set of endowments is given by

$$\Omega = \left\{\omega^l, \omega^h\right\}, \qquad \frac{\pi^{l,h}}{\pi^{h,l} + \pi^{l,h}}\omega^h + \frac{\pi^{h,l}}{\pi^{h,l} + \pi^{l,h}}\omega^l = 1.$$

Total aggregate endowment each period is split between transfers to the young and middle-aged individuals in the economy and a fruit from a tree, where the former can be thought of corresponding to labor income, whereas the latter can be thought of as capital income.

$$w_y^s + w_m^s + \delta^s = \omega^s \qquad s \in \{l, h\}$$

2.2.1 Competitive Equilibrium

As in the benchmark model agents receive an age and state dependent income when young (w_y) and middle-aged (w_m). They have two assets available: bonds which are in zero net supply and pay deterministically one unit of consumption next period, and equity, which is in fixed supply, normalized to one. Each period a dividend δ is paid out to the equity holders. The dividend process follows a two-stage Markov process.

Each agent maximizes discounted life-time expected utility conditional on the state in which the agent is born.

$$\mathrm{E}\left(U\right) = u\left(c_y\right) + \beta\mathrm{E}\left[u\left(c_m\right)\right] + \beta^2\mathrm{E}\left[u\left(c_r\right)\right]$$

subject to the individuals' period-by-period budget constraints

$$c_y = w_y - qb_y - pe_y$$
$$c_m = w_m + b_m + (p + \delta)\,e_y - qb_m - pe_m$$
$$c_r = w_r + b_m + (p + \delta)\,e_m$$

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As before, $u(\cdot)$ is a strictly increasing, strictly concave function that satisfies the Inada conditions, and q and p are the prices of equities and bonds.

The market clearing conditions are

$$b_y + b_m = 0,$$
$$e_y + e_m = 1,$$

and the overall resource constraint

$$c_y + c_m + c_r \leq w_y + w_m + \delta = \omega.$$

In any period, young and middle aged individuals will solve their optimization problem. The first order condition with respect to the two assets for the young agents are

$$-qu'(c_y) + \beta\mathrm{E}\left[u'(c_m)\right] = 0$$
$$-pu'(c_y) + \beta\mathrm{E}\left[(p' + \delta) u'(c_m)\right] = 0$$
$$-qu'(c_m) + \beta\mathrm{E}\left[u'(c_r)\right] = 0$$
$$-pu'(c_m) + \beta\mathrm{E}\left[(p' + \delta) u'(c_r)\right] = 0$$

One can show that the competitive equilibrium in a stochastic overlapping generations models with either more than one good, or in which agents live more than two periods is generically not strongly stationary, in the sense that equilibrium prices and allocations depend only on the current realization of the exogenous state variable[1]. In these models, the best we can hope for is that there is a recursive representation of equilibrium in which prices and allocations depend on the current state of nature and a set of lagged endogenous variables.

At any point in time, the economy is characterized by the realization of the endowment process (ω), lagged bond holdings and equity holdings by the middle aged $(b_{y,-1}, e_{y,-1})$ The vector $s \in S \subset \mathbb{R}^3$ is the state of the economy. Hence, an equilibrium is a sequence of allocations $\{b^y(s), e^y(s)\}$ and a sequence of prices $\{q(s), p(s)\}$ such that

1. Each individual solves her/his optimization problem subject to budget constraint

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[1]This result was first shown by Spear (1985) for OLG economies in which agents live two periods but trade multiple goods. To the best of our knowledge, the corresponding result for single good economies in which agents live more than two periods was first shown by Aiyagari while he was visiting Carnegie Mellon in 1984, though the result does not appear to have been published anywhere.

2. The bond and equity markets clear and the aggregate resource constraint holds.

3 Optimality in the Stochastic Economy

We turn now to the main part of the paper, and examine the competitive equilibrium for a model with aggregate shocks to total resources laid out in Section 2, and examine the risk-sharing properties and related optimality issues of this equilibrium.

Our first result characterizes perfect risk sharing.

Theorem 2. *Perfect risk sharing implies a strongly stationary consumption sequence.*

Proof. The optimal stationary risk-sharing allocation will be a solution to the optimization problem

$$\max_{f} \sum_{i=y,m,r} \gamma^i E_i u_i \left(f_i \right)$$

subject to

$$f_y \left(\omega^s, \mathbf{z} \right) + f_m \left(\omega^s, \mathbf{z} \right) + f_r \left(\omega^s, \mathbf{z} \right) = \omega^s$$

where each agent's allocation is assumed to be a function of current resources ω^s, and lagged endogenous variables, which we denote by z. We index the expectation operator by each agent's type to allow for different notions of optimality. The different notions we consider are *ex ante* optimality across all agents, resource state conditional *ex interim* optimality (in which the planner considers takes the expectation for young agents over lagged endogenous variables, but not over the resource state in which the young were born), and conditional optimality (in which the planner takes the current resource state and realized lagged endogenous variables of the young as given). Letting μ be the (assumed known) invariant distribution of lagged endogenous variables, the expected utility for each middle-aged or retired agent in the optimization problem for any optimality concept is given by

$$\int \left[\pi^l u_i \left(f_i \left[\omega^l, z \right] \right) + \pi^h u_i \left(f_i \left[\omega^h, z \right] \right) \right] du \left(z \right) \quad \text{for } i = m, r.$$

For an *ex ante* notion of optimality, this will also be the expected utility of the young. In the resource conditional *ex interim* case, the expected utility of the young will be

$$\int u_y \left(f_y \left[\omega^s, z \right] \right) d\mu \left(z \right)$$

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while in the conditional case, the objective function for the young will simply be $u_y\left(f_y\left[\omega^s, z\right]\right)$. For the *ex ante* and resource conditional *ex interim* cases, the first-order conditions for the optimization are

$$\gamma^i \pi^s u_i'\left(f_i\right) g\left(z\right) - \lambda^s = 0, \text{ for } s = l, h \text{ and } i = y, m, r$$
$$f_y\left(\omega^s, \mathbf{z}\right) + f_m\left(\omega^s, \mathbf{z}\right) + f_r\left(\omega^s, \mathbf{z}\right) = \omega^s$$

where $g\left(z\right)$ is the Radon-Nikodym derivative of the measure μ with respect to z. These conditions imply that

$$\gamma^i \pi^s u_i'\left(f_i\right) = \gamma^j \pi^s u_j'\left(f_j\right) \text{ for } s = l, h \text{ and } i \neq j.$$

Hence, we can solve for say f_y and f_m in terms of f_r. Substituting back into the resource constraints will then yield allocations which are strongly stationary. Taking ratios in the first set of first-order conditions, we get the usual equality of state contingent marginal rates of substitution condition:

$$\frac{\pi^l u_i'\left(f_i\left[\omega^l, z\right]\right)}{\pi^h u_i'\left(f_i\left[\omega^h, z\right]\right)} = \frac{\lambda^l}{\lambda^h} \text{ for } i = y, m, r.$$

Note in particular that the probability $g(z)$ for lagged endogenous state variables drops out, since it is the same across current realizations of total resources. For the conditional optimality case, the first-order conditions for the middle-aged and retired are as above, while for the young, they become

$$\gamma^y u_y'\left(f_y\right) - \lambda^s = 0.$$

In this case, since the Lagrange multiplier associated with the resource constaint doesn't depend on lagged endogenouse variables, the allocation of the young will also be independent of these variables. Since the first-order conditions of the middle-age and retired remain as before, their allocations are also independent of lagged endogenous variables, and we again obtain the result that optimal allocations are strongly stationary. $\square$

This result is quite intuitive. Since the exogenous uncertainty is independent of any endogenous uncertainty, the optimal allocation simply ignores endogenous fluctuations, and allocates total resources in a way that minimizes the variance associated with these fluctuations. There is one additional curious possibility that we might need to consider, which is that the weights in the social planner's problem might themselves be functions of lagged endogenous variables. Since the first-order conditions require that

$$\gamma_y \pi^s u_y'\left(f_y\right) = \gamma_m \pi^s u_m'\left(f_m\right) = \gamma_r \pi^s u_r'\left(f_r\right) \text{ for } s = l, h$$

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this would yield allocations depending on the lagged endogenous variables. But in this case, since the planner is free to choose the weights, *ex ante* optimization would lead him to choose constant weights, since these minimize the variances of each agent's allocation. Note finally that this result can be extended readily to Markovian shock processes.

This result tells us immediately that the competitive equilibrium in a stochastic OLG economy in which agents live for more than two periods or consume more than a single good each period does not allocate risk optimally. This does not imply, however, that the competitive equilibrium is not Pareto optimal. To see this, consider a stochastic economy in which agents live two periods and consume $\ell > 1$ goods per period. Suppose that endowments of the young are stochastic on a two point support: $\left[\omega^L, \omega^H\right]$. We will denote the states as low and high, and assume that $\omega^H \gg \omega^L$. We know that for this economy, the competitive equilibrium cannot be strongly stationary, and hence, risk is allocated suboptimally. Suppose at some time t we try to reallocate consumptions in such a way as to reduce the variance of the old agent's consumption at time $t+1$. This requires that we increase the old agent's consumption in the low state while decreasing it in the high state. Doing this will make a young agent born in the high state better off, but will make the young agent born in the low state worse off. Hence, we can't improve risk sharing in this setting, and the competitive equilibrium is, in fact, Pareto optimal.

We turn next to an analysis of the socially optimal stationary equilibrium in the stochastic version of our benchmark model, and examine two distinct notions of optimality, following Demange (2002). The first is the notion of stationary *ex ante* optimality, where we allow the social planner to choose allocations for unborn generations which maximize their expected utility. This is in contrast to *interim* or *conditional* optimality, in which the social planner is constrained to treat agents born in different states of nature as if they were different agents.

The social planner's problem in the *ex ante* case is

$$\max_{(c^y, c^m, c^r)} Eu\left(c^y\right) + \beta Eu\left(c^m\right) + \beta^2 Eu\left(c^r\right)$$

subject to

$$c_s^y + c_s^m + c_s^r = w^y + w^m + \delta^s, \ \ s = l, h.$$

Since

$$Eu\left(c_i\right) = \pi^{sl} u\left(c_i^s\right) + \pi^{sh}\left(c_i^s\right)$$

for $i = y, m, r$ and $s \in \{l, h\}$ the resource state in the previous period, it

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follows that the first-order conditions for the social planner's problem are

$$u'\left(c_y^s\right) - \lambda^s = 0, \ \text{for } s = l, h$$
$$\beta u'\left(c_m^s\right) - \lambda^s = 0, \ \text{for } s = l, h$$
$$\beta^2 u'\left(c_r^s\right) - \lambda^s = 0, \ \text{for } s = l, h$$

together with the resource constraints for the two possible state realizations. Since the optimal allocations clearly won't depend on any lagged realizations of the state or on lagged allocations, they are strongly stationary.

Now, in the benchmark deterministic version of the model, we saw that at the socially optimal steady-state, the intertemporal marginal rate of substitution was 1. The following proposition generalizes this result to the stochastic model.

Proposition 3. *For any two-stage Markov process, in the limit, consumption is optimally distributed such that*

$$\beta \frac{u'\left(c_s^m\right)}{u'\left(c_s^y\right)} = \beta \frac{u'\left(c_s^r\right)}{u'\left(c_s^m\right)} = 1 \qquad s \in \{l, h\}$$

given the resource constraints

$$c_s^y + c_s^m + c_s^r = \omega_s.$$

Proof. The social planner faces the following optimization problem

$$
\max_{\left\{c_y^l, c_m^l, c_r^l, c_y^h, c_m^h, c_r^h\right\}} \left\{ \Pi_l \left[u\left(c_y^l\right) + \beta \left[\pi_{ll} u\left(c_m^l\right) + \pi_{lh} u\left(c_m^h\right)\right] \right. \right.
$$
$$
\left. + \beta^2 \left[\pi_{ll}\pi_{ll} u\left(c_r^l\right) + \pi_{ll}\pi_{lh} u\left(c_r^h\right) + \pi_{lh}\pi_{hl} u\left(c_r^l\right) + \pi_{lh}\pi_{hh} u\left(c_r^h\right)\right]\right]
$$
$$
+ \Pi_h \left[u\left(c_y^h\right) + \beta \left[\pi_{hl} u\left(c_m^l\right) + \pi_{hh} u\left(c_m^h\right)\right]\right.
$$
$$
\left. \left. + \beta^2 \left[\pi_{hl}\pi_{ll} u\left(c_r^l\right) + \pi_{hl}\pi_{lh} u\left(c_r^h\right) + \pi_{hh}\pi_{hl} u\left(c_r^l\right) + \pi_{hh}\pi_{hh} u\left(c_r^h\right)\right]\right] \right\},
$$

where $\{\Pi_h, \Pi_l\}$ is the limit distribution of the Markov process. The resource constraints are

$$
\lambda \ : \ c_y^l + c_m^l + c_r^l = \omega^l
$$
$$
\mu \ : \ c_y^h + c_m^h + c_r^h = \omega^h
$$

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The intertemporal rates of substitution can be expressed as:

$$\beta \frac{u'\left(c_m^l\right)}{u'\left(c_y^l\right)} = \frac{\Pi_l}{\Pi_l \pi_{ll} + \Pi_h \pi_{hl}}$$

$$\beta \frac{u'\left(c_r^l\right)}{u'\left(c_m^l\right)} = \frac{\Pi_l \pi_{ll} + \Pi_h \pi_{hl}}{\Pi_l \left(\pi_{ll}\pi_{ll} + \pi_{lh}\pi_{hl}\right) + \Pi_h \left(\pi_{hl}\pi_{ll} + \pi_{hh}\pi_{hl}\right)}$$

$$\vdots$$

All the right hand side expressions are equal to 1 and the result follows. $\square$

4   Numerical simulations

The numerical simulations are meant to illustrate our theoretical results, to give an indication of the quantitative importance of the suboptimality of the C.E., and to give an estimate for the time necessary to reach the stationary, optimal equilibrium.

4.1   Parametrization

As a benchmark, we work with a parametrization of the model in which both the dividend and labor income are stochastic. Specifically, we assume that labor's share of the total endowment is $\frac{2}{3}$, and the ratio of labor income when young to labor income when middle-aged is $\frac{3}{5}$. So, for any given total endowment ω, we have

$$\delta = \frac{1}{3}\omega,$$

$$w_y = \frac{3}{8}\frac{2}{3}\omega = \frac{1}{4}\omega,$$

$$w_m = \frac{5}{8}\frac{2}{3}\omega = \frac{5}{12}\omega.$$

The endowment-process follows a two-stage Markov process. For the benchmark simulations, the endowment process is assumed to be i.i.d. with realizations $\left\{\omega^l, \omega^h\right\} = \{0.95, 1.05\}$.

The time-preference parameter (β) is equal to 1 and the risk aversion coefficient is set equal to 2.

4.2   Numerical results

For these parameters, the socially optimal allocation gives each agent 0.35 units of consumption in the high state, and 0.317 units in the low state.

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Table 1: Summary statistics, benchmark

	c_y	c_m	c_r
Low	0.19719	0.30268	0.45013
(SD)	0.00018	0.00099	0.00117
High	0.21388	0.32490	0.51122
(SD)	0.00020	0.00112	0.00131

The expected utility for the social optimum is $Eu = -9.02$. The standard deviation of the social optimum is 0.0233.

At the competitive allocation, Table 1 shows the average consumptions and standard deviations of consumption for the various types of agents in the high and low states

The expected utility for a typical agent can be estimated from the simulation data using the fact that the equilibrium allocations follow an ergodic stochastic process, so that time-series and cross-sectional averages will be the same. From this data, we find that $Eu = -10.14$. Hence, the socially optimal allocation improves overall expected utility for this calibration of the model by roughly 11%. The overall average standard deviations of consumption for each type of agent are $\sigma_y = 0.008$, $\sigma_m = 0.011$, and $\sigma_r = 0.031$. Clearly, the old bear far more risk in the competitive equilibrium than they do at the social optimum.

We turn next to the question of full Pareto optimality. We have seen that the socially optimal stochastic steady-state does allocate risk efficiently, but this leaves open the question (as we saw in the example above) of whether this allocation actually dynamically Pareto dominates the competitive equilibrium. On this issue, we have the following result.

Theorem 4. *If agents live more than two periods, the laissez-faire competitive equilibrium allocation is not dynamically Pareto optimal.*

Proof. Since the competitive equilibrium is not strongly stationary, risk is not allocated optimally. This means that in any given period t, some agents will face contingent allocations in the following period such that their state-contingent marginal rates of substitution at these allocations are not equal. These agents can generate an *ex ante* Pareto improvement by having the agents with the smaller MRS move along her indifference curve toward the diagonal (reducing variance while keeping expected utility constant), while the agent with the larger MRS takes the opposite side of this swap. As we

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illustrated earlier, this makes the agent with the higher MRS strictly better off. We can obviously also find an reallocation between these two agents which makes them both strictly better off. This reallocation also clearly makes no other agent in the economy worse off, so it constitutes a short-run Pareto improvement, and hence, the competitive equilibrium is not short-run optimal. Since short-run optimality is necessary for long-run optimality, it follows that the competitive equilibrium is not Pareto optimal. $\qquad\square$

We illustrate these results in our numerical simulation by first showing directly that the kind of improved risk-sharing described in the theorem is indeed possible in the simulated economy. The diagram below shows the tree of realized consumptions for middle-aged and old agents over a span of four periods. In the tree, branches going up indicate realizations of the high resource state, while branches going down indicate a low resource state realization. For each allocation at each node of the tree, we calculate the marginal rate of substitution between low and high state allocations for both middle-aged and old, and, as is apparent from the diagram, the middle-aged have uniformly higher MRS's than do the old, indicating that there is always a risk-improving reallocation of the type indicated above that is possible.

We can generalize this direct demonstration to show that such improvements are always possible using the fact that the distribution of allocations generated by the competitive equilibrium is ergodic, and hence time averages and cross-sectional averages will be the same. Consider a middle-aged and a retired agent at any time t. For the parametrization of the model, the state-contingent MRS for the middle-aged agent will be

$$MRS_m^{hl} = \left(\frac{c_m^l}{c_m^h}\right)^2$$

while that of the old agent will be

$$MRS_r^{kl} = \left(\frac{c_r^l}{c_r^h}\right)^2$$

where c_j^i is the $j = H, L$ state allocation of agent $i = m, r$. Using the resource constraint, we can write the old agent's MRS as

$$MRS_r^{hl} = \left(\frac{r^l - c_y^l - c_m^l}{r^h - c_y^h - c_m^h}\right)^2$$

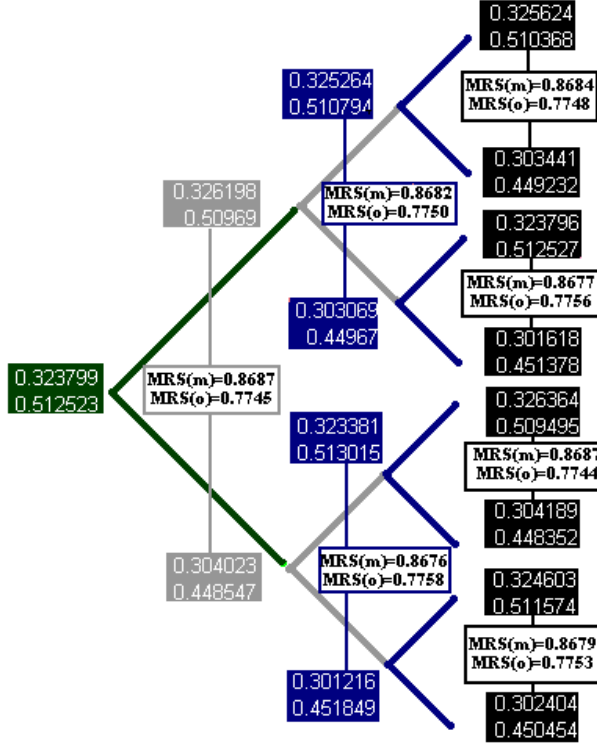$$= \left(\frac{\hat{r}^l - c_m^l}{\hat{r}^h - c_m^h}\right)^2$$

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Figure 1: Tree realizations

where $\hat{r}^s$ is the total state s resources net of the allocation of the young. Now, consider the middle-aged agent's share of net resources in the low state

$$\frac{c_m^l}{\hat{r}^l}$$

and suppose that this agent gets the same share of net resources when the state is high. In this case, her MRS will be

$$MRS_m^{lh} = \left(\frac{c_m^l}{\left(\frac{c_m^l}{\hat{r}^l} \hat{r}^h \right)} \right)^2 = \left(\frac{\hat{r}^l}{\hat{r}^h} \right)^2.$$

Since the old agent's share in the high state in this case is $1 - \frac{c_m^l}{\hat{r}^l} = \frac{c_r^l}{\hat{r}^l}$, it

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follows that the old agents state-contingent MRS will be

$$MRS_r^{lh} = \left(\frac{c_r^l}{\left(\frac{c_r^l}{\hat{r}^l}\hat{r}^h\right)}\right)^2 = \left(\frac{\hat{r}^l}{\hat{r}^h}\right)^2$$

and hence, since the MRS's are equal, we would have optimal risk-sharing. Since we know that the CE does not allocate risk optimally, it must be the case that one agent's share of the net resources in, say, the high state is greater than the other agent's. So, suppose the middle-aged agent gets a smaller share in the high state than she does in the low state. Let $0 < \varepsilon < 1$ and assume that her high state share is

$$(1 - \varepsilon)\frac{c_m^l}{\hat{r}^l}.$$

Then the old agent's high state share will be

$$(1 + \varepsilon)\frac{c_r^l}{\hat{r}^l}.$$

Plugging these into the MRS formulas, we have

$$MRS_m^{lh} = \left(\frac{c_m^l}{(1-\varepsilon)\frac{c_m^l}{\hat{r}^l}\hat{r}^h}\right)^2 = \left(\frac{\hat{r}^l}{(1-\varepsilon)\hat{r}^h}\right)^2$$

and

$$MRS_r^{lh} = \left(\frac{c_r^l}{(1+\varepsilon)\frac{c_r^l}{\hat{r}^l}\hat{r}^h}\right)^2 = \left(\frac{\hat{r}^l}{(1+\varepsilon)\hat{r}^h}\right)^2$$

and the old agent will have a smaller state-contingent MRS than the middle-aged.

The simulated data show that these kinds of opportunities for improved risk-sharing always exist. Figure 2 shows the shares for old and middle-aged in each of the resource states for a simulated time-series (after convergence of the numerical algorithm used to compute the equilibrium) consisting of 3000 periods. Clearly, the old get a uniformly higher share of net resources in the high state, while the middle-aged get a higher share in the bad state. Hence, via the argument above, there are always opportunities for improved risk-sharing.
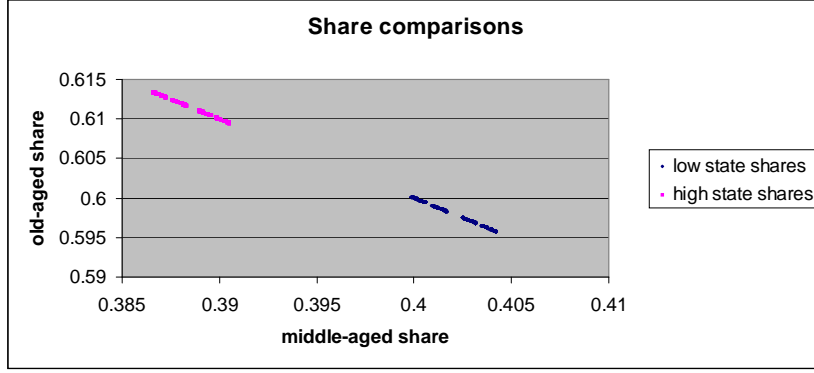
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Figure 2: Share comparisons

5   Transition to the optimal allocation

We have now established that the weakly stationary sequence of allocations resulting from the recursive stochastic equilibrium is dynamically not optimal. In any overlapping generations setting, the essence of the dynamic suboptimality concept is that it is then possible to Pareto improve and move to optimal one.

The hard – and interesting – question is whether from any state $s \in \mathbb{R}^3$ it is possible to construct a sequence which would make every individual *conditionally* weakly better off, at least one individual strictly better off, and which would end up in the *ex ante* strongly stationary optimal allocation as described in Proposition 3.

In order to formalize the transition, define the central planner's welfare function as of time τ, given a partial history $h_t^\tau = \left[\left(\hat{c}_y^j, \tilde{c}_m^j, \hat{c}_r^j \right)_{j=t}^\tau \right]$, where $\hat{c}_i^j$, $i = y, m, r$ is a competitive equilibrium consumption for an agent if type i at time j

$$W \left(h_t^{\tau-1} \right) \max_{c^y, c^m, c^r} \mathrm{E} \left[u \left(c_y^\tau \right) \right] + \beta \mathrm{E} \left[u \left(c_m^\tau \right) \right] + \beta^2 \mathrm{E} \left[u \left(c_r^\tau \right) \right]$$

subject to reservation utilities and resource constraints.

$$u \left(c_y^\tau \right) \geq u \left(\tilde{c}_y^\tau \right)$$
$$\beta \mathrm{E} \left[u \left(c_m^\tau \right) \right] \geq \mathrm{E} \left[U^{\tau-1} \mid h_t^{\tau-1} \right] - u \left(c_y^{\tau-1} \right) - \beta^2 \mathrm{E} \left[u \left(c_r^{\tau+1} \right) \mid h_t^{\tau-1} \right]$$
$$\beta^2 \mathrm{E} \left[u \left(c_r^\tau \right) \right] \geq \mathrm{E} \left[U^{\tau-2} \mid h_t^{\tau-2} \right] - u \left(c_y^{\tau-2} \right) - \beta \mathrm{E} \left[u \left(c_m^{\tau-1} \right) \mid h_t^{\tau-1} \right]$$
$$\omega \geq c_y^\tau + c_m^\tau + c_r^\tau$$

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Define a function $P : \mathbb{R}^{3(T-t)}_{++} \to \mathbb{R}$ as the sum of the expected utilities of all individuals alive between time t and time T

$$P\left(c_y^t, c_m^t, c_r^t, \ldots, c_y^t, c_m^t, c_r^T\right) = \sum_t \mathrm{E}\left[u\left(c_y^t\right)\right] + \beta \mathrm{E}\left[u\left(c_m^{t+1}\right)\right] + \beta^2 \mathrm{E}\left[u\left(c_r^{t+2}\right)\right]$$

such that for every time t and every allocation $c_{-t} \in \mathbb{R}^{N-1}_{++}$

$$W\left(c_\tau, h_t^{\tau-1}\right) - W\left(x_\tau, h_t^{\tau-1}\right) > 0 \quad \text{iff} \quad P\left(c_\tau, h_t^{\tau-1}\right) - P\left(x_\tau, h_t^{\tau-1}\right) > 0$$

A function P satisfying the last equation is called an ordinal potential, and a game that possesses a ordinal potential is called a ordinal potential game.[2]

In order to implement this algorithm, we proceed with the following steps:

- Given the decision rules of the agents and the laws of motions of prices and aggregates, we take an arbitrary point in the state space $\left(\omega, b_y^{-1}, e_y^{-1}\right)$, generate all possible sequences of length N of the the realizations of the exogenous variable. These sequences of realizations can be ordered in a binary tree with $\sum_{n=1}^{N} 2^{n-1}$ nodes.

- For each realization of the exogenous variable, given the state and the the individuals' decision rules, we calculate the consumption-triplet for young, middle-aged and old.

- Looking ahead one period at the child nodes n_1, n_2 of any node n, we impose optimal risk-sharing on the middle-aged and old at the child nodes (i.e. the young and middle-aged at node n), subject to the condition that the old get at least the same expected utility as they would have at the competitive equilibrium. Since this generates a strict Pareto improvement for the node n young, we transfer consumption on the the four grandchild nodes $n_{11}, n_{12}, n_{21}, n_{22}$ nodes from the old to the grandchild node young, making them strictly better off.

- Moving ahead to node n_i, $i = 1, 2$, we repeat the process of imposing optimal risk-sharing between middle-aged and old on subsequent nodes, and transferring resources to the young on the grandchild nodes.

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[2]Monderer and Shapley (1996) proved several properties of potential games that might be helpful for us, like: equilibrium refinement, *finite improvement property*, and that the entire sequence of allocations might be interpreted as a sequence of Nash equilibria. The usefulness of the potential function is not just that it is (locally) maximized at a Nash equilibrium. It also provides a direct tool to prove equilibrium stability under the directional adjustment hypothesis in (2). Indeed, in the absence of noise, the potential function itself is a Liapunov function.
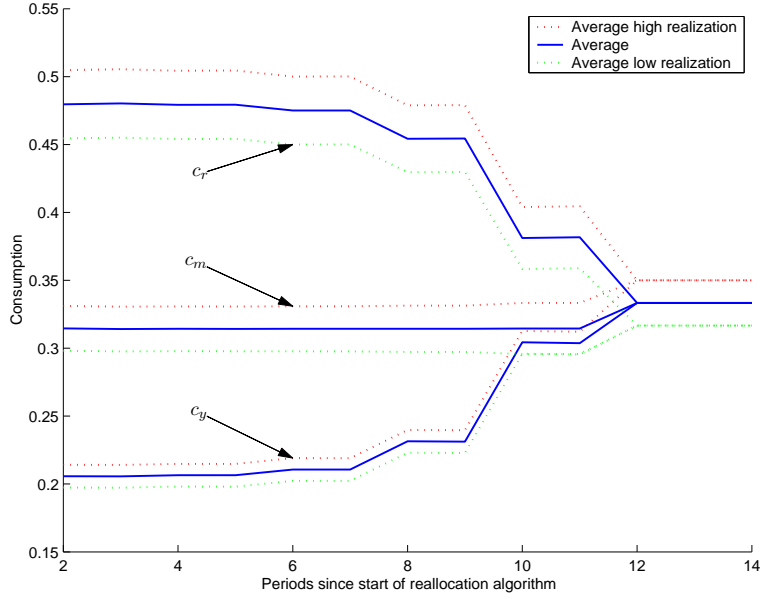
Figure 3: Consumption through the transition

- This process terminates when we reach the social optimum, which for our parametrization of the model economy has all agents consuming $\hat{c}^l = \omega^l/3 = 0.317$ and $\hat{c}^H = \omega^H = 0.35$.

The diagram below illustrates how the algorithm attains the strongly stationary social optimum in a finite number of steps. Given the way we have defined the reallocation algorithm, the number of periods it takes to attain the social optimum is the same along all branches of the binary tree.

The next diagram shows the same process, but in terms of the changes in expected utility. For this construction, we hold the overall expected discounted utility at birth constant at the competitive equilibrium values during the reallocation process, up to the point where it becomes possible to strictly dominate the (now strongly stationary) allocation. As in the previous diagram, the dotted lines are the average expected utilities given the exogenous resource state, while the solid line is the overall average expected utility.

5.1 Decentralized optimality

Since the socially optimal allocation for the economy is a Pareto optimal steady-state, the second welfare theorem implies that it can be supported
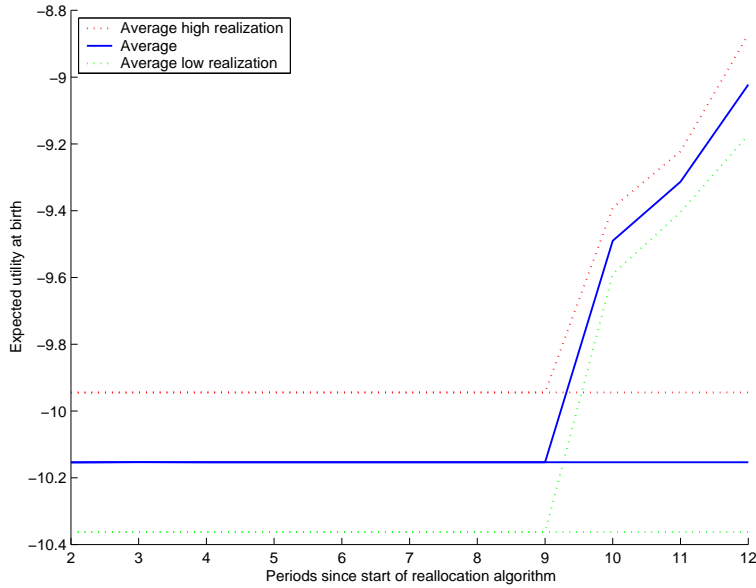
Figure 4: Discounted utility through the transition

as a competitive equilibrium after some reallocation of resources. Since we know that with the productive asset pays any positive dividend, the resulting competitive equilibrium will not be strongly stationary and hence not optimal, it follows that to implement the optimal steady-state, the central planner (which we will refer to hereafter as the government) must completely tax away the dividend, and then redistribute it back to the agents as lump sum transfers. One way the government could implement the optimal steady-state would be to give agents exactly the right shares of total resources so as to implement the optimal allocation as a no-trade equilibrium. While this is easy enough to do in our simple three-period setting, in more complicated economies, it would require the government to hit an allocation target having measure zero in the space of all possible state contingent allocations. So, we inquire instead whether, having taxed away the dividend and then rebated it arbitrarily, agents can then trade via competitive markets to the optimal allocation. The answer to this question is yes, if we introduce a set of Arrow securities to allow agents to insure themselves against the intertemporal effects of the resource shocks.

To show this result, we denote the post-transfer endowments of agents as $\tilde{\omega}_i^s$, for $i = y, m, r$ and $s = h, l$. We assume that agents can (as before) trade one period bonds in zero net supply in order to allocate income in-

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tertemporally. We also introduce a set of Arrow securities, denoting the holdings of a type i agent born in resource state s which pays off in one unit of consumption in state s' by $a^{ss'}$. We will denote the bond prices by q^s, as before, and let p^s be the (nominal) price of the Arrow security that pays off in state s. With these modifications, the budget constraints for the model (under the assumption that we are at a strongly stationary allocation) take the form

$$c_y^s = \tilde{\omega}_y^s - q^s b_y^s \quad \text{for } s = h, l$$

$$\sum_{s'} p^{s'} a_y^{ss'} = 0 \quad \text{for } s = h, l$$

$$c_m^{ss'} = \tilde{\omega}_m^{s'} + b_y^s + a_y^{ss'} - q^{s'} b_m^s \quad \text{for } (s, s') \in \{h, l\}^2$$

$$\sum_{s''} p^{s''} a_m^{s's''} = 0 \quad \text{for } s' = h, l$$

$$c_r^{s's''} = b_m^{s'} + a_m^{s's''} \quad \text{for } (s', s'') \in \{h, l\}^2.$$

The constraint on the Arrow securities requires that they be self-financing in the sense of allowing transfers only between different states of nature and not over time. While this constraint is not necessary, it helps clarify the different functions of the two types of assets.

The first-order conditions for the budget constrained utility maximizations of young and middle-aged agents are

$$-q^s u'\left(c_y^s\right) + \sum_{s'} \pi^{s'} u'(c_m^{ss'}) = 0 \quad \text{for } s = h, l$$

$$u'\left(c_m^{ss'}\right) - \lambda^s p^{s'} = 0 \quad \text{for } (s, s') \in \{h, l\}^2$$

$$-q^{s'} u'\left(c_m^{ss'}\right) + \sum_{s''} \pi^{s''} u'(c_r^{s's''}) = 0 \quad \text{for } (s', s'') \in \{h, l\}^2$$

$$u'\left(c_r^{s's''}\right) - \lambda^{s'} p^{s''} = 0 \quad \text{for } (s', s'') \in \{h, l\}^2$$

together with the budget constraints above. Here, λ^s is the Lagrange multiplier associated with the self-financing condition on the Arrow securities. Market clearing for the model requires that

$$c_y^s + c_m^{s's} + c_r^{s's} = \tilde{\omega}_y^s + \tilde{\omega}_m^s + \tilde{\omega}_r^s \quad \text{for } s = h, l$$

$$b_y^s + b_m^s = 0 \quad \text{for } s = h, l$$

$$a_y^{s's} + a_m^{s's} = 0 \quad \text{for } (s, s') \in \{h, l\}^2.$$

Now, the first-order condition for the middle-aged agent implies that consumption of the middle-aged at equilibrium can't depend on the agent's birth state, since the right-hand expected utility of consumption when old for the agent doesn't depend on the birth state. Together with the resource constraints, this implies that consumption when old only depends on the current resource state. The first-order conditions for the Arrow securities then imply that the Lagrange multipliers are independent of the lagged shock. Note also that the consumption market-clearing conditions are redundant given asset market-clearing and the budget constraints. Finally, the first-order conditions for the Arrow securities implies that

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{p^h}{p^l}$$

while the first-order condition with respect to the middle-aged agent's bond holdings implies that

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{q^l}{q^h}$$

so that the bond and Arrow security prices are not independent.

Hence, we are left then with a system of 14 equations in 14 variables: the 2 bond prices, 4 bond holdings, 8 Arrow security holdings. These equations can be solved under standard conditions using standard techniques.

To show that the competitive equilibrium allocation allocates risk optimally, consider the first-order conditions for the Arrow security holdings for a middle aged agent. Fixing s and taking $s' = h, l$, and taking ratios we get

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{p^h}{p^l}.$$

Doing the same thing for the old agent, we find

$$\frac{u'\left(c_r^h\right)}{u'\left(c_r^l\right)} = \frac{p^h}{p^l}.$$

Hence, the middle-aged and old have their state-contingent marginal rates of substitution equalized and are sharing risk optimally. The first-order conditions with respect to bond holdings, together with the price dependence between bond and Arrow security prices show that the state-contingent MRS of the young is also equal to p^h/p^l.

To show finally that the socially optimal allocation will be attained in this setting, we need to make one more assumption: the government transfer

to the old is not so large that the endowment allocation is already optimal. We need this assumption since it's possible that if the transfer to the old is large, the endowment allocation will be in the classical region. Given this assumption, we can determine the competitive equilibrium prices which support the social optimum by plugging in the consumptions at the social optimum in the equilibrium equations above, and backing out the prices.

There are a couple of important observations about this result that we should make. First, the result shows in particularly stark form the importance of providing some form of intergenerational insurance if we are interested in agents' overall welfare. It is clear from the budget constraints, particularly for the old agents, that the Arrow securities allow the agent to smooth consumption across shock realizations in ways that the bond holdings alone do not. This suggests an obvious role for government insurance programs in the absence of private provision of assets of this form.

The second observation concerns the fact that we can find a strongly stationary equilibrium with Arrow securities, but not with privately held productive assets. We believe the reason for this is that the productive asset is in positive net supply, and must be held (i.e. owned) in order for the benefit of the dividend to be realized. Think of allowing the price of the asset to depend on the previous period's resource shock realization, as the Arrow security price does. This will allow middle-aged and old agents to use the productive asset to hedge risk associated with bond prices, but it also means that the price the young must pay to acquire the asset now depends on the resource state in the period prior to their birth. Since they must hold this asset to realize any gain from it, this immediately introduces a history dependence into the equilibrium prices which precludes it from being strongly stationary, and hence, from allocating risk optimally.

While we have demonstrated how the second welfare theorem can be applied, the requirement that the government essentially confiscate the full return on the assets is obviously problematic. To the extent that asset returns provide incentives in more complicated economic environments in which agents make investments in productive activities, removing this incentive by taxing it completely away will cause obvious problems. So, we look next at an alternative to the full application of the second welfare theorem, by considering a tax on dividends that is significantly less than confiscatory, coupled with a lump sum rebate of the tax which moves the economy in the "right direction" toward the social optimum, and ask whether this will implement a set of Pareto improving allocations.

5.1.1 Stochastic steady state with taxes and transfers

Kubler and Krueger (2002) showed via a numerical example that a pay-as-you-go social security system can lead to Pareto improvements after bad state realizations in a model similar to ours, so there is reason to think this will also be so here. The model with taxes and transfers is the same as our benchmark model, except that we modify the budget constraints to reflect the fact that the government now imposes a proportional tax of t on the dividend realization, and uses the proceeds from this tax to give lump sum transfers τ^i, $i = y, m, r$ to the households. Hence, the sequential budget constraints become

$$c_y \leq \omega_y - qb_y - pe_y + \tau_y$$
$$c_m \leq \omega_m + b_y + e_y \left[p' + (1-t)\delta \right] - q'b_m - p'e_m + \tau_m$$
$$c_r \leq b_m + e_m \left[p'' + (1-t)\delta \right] + \tau_r.$$

The government's budget constraint requires that in each period

$$\tau_y + \tau_m + \tau_r = t\delta.$$

We compute the recursive competitive equilibrium of the tax-transfer model for tax rates of 25%, 50% and 75%, under the assumption that all of the lump sum transfer goes to the old. The results for each tax level clearly improve risk-sharing and raise expected utility. The table below shows the variance of consumption for the old, the overall expected utility, and minimum consumption for the young at each tax rate.

Table 2: Summary statistics, benchmark

Tax rate	var (c_r)	min (c_y)	EU
0.00	0.000934	0.19692	-10.14377318
0.25	0.000859	0.2053	-9.963402732
0.50	0.000770	0.21629	-9.75796138
0.75	0.000645	0.23241	-9.51129027

The chart below plots the consumptions of the young in the zero tax case and the 75% tax case over a typical run of 100 periods. From the diagram, it is clear that the young will be getting strictly more consumption under the tax-transfer equilibrium than in the zero tax case. Combining this

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observation with the expected utility improvement, we infer that the tax-transfer equilibrium will in fact Pareto dominate the zero tax equilibrium.
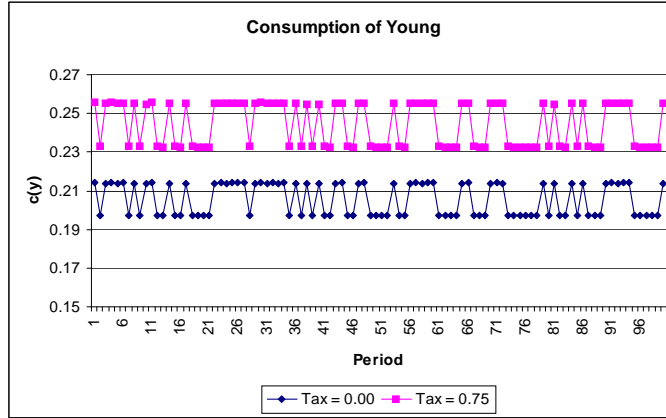

Figure 5: Consumption of young individuals at different tax rates

6 Stochastic fertility

One obvious extension of the model we have been examining is to allow for varying population size. Indeed, if we take the 20 year time span of each period of the model seriously, over these time spans, it is likely that households will face risk from variations in population fertility rates, which lead to larger or smaller cohorts entering into the economy.

This extension of the model parallels the model examined by Geanako-plos, Magill and Quinzii (2004). For this version of the model, we assume that the size of the young cohort born at any point in time can take on one of two values, n or N, with $n < N$. Population states follow a Markov process, and, for simplicity, we will assume that for all transitions, the large and small cohort sizes are equally probable. The eight possible population

states for the economy are

- State 0 : $\{n, n, n\}$
- State 1 : $\{n, n, N\}$
- State 2 : $\{n, N, n\}$
- State 3 : $\{n, N, N\}$
- State 4 : $\{N, n, n\}$
- State 5 : $\{N, n, N\}$
- State 6 : $\{N, N, n\}$
- State 7 : $\{N, N, N\}$

where the first entry is the size of a newly born young cohort, followed by the middle aged cohort and then the old. Figure 6 shows the possible transitions between population states.
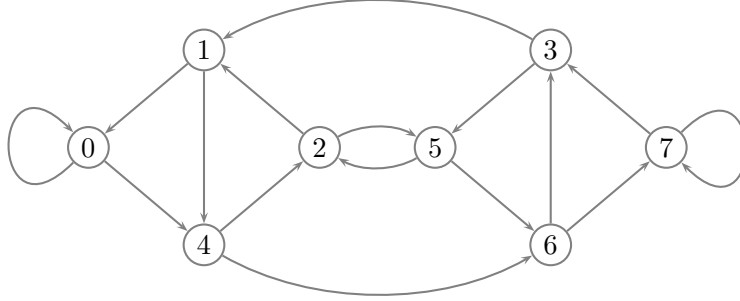

Figure 6: Markov population state transitions

Except for the variable population sizes, the model is the same as in the previous section. The only difference is that now, the resource constraint takes the form

$$\eta_t c_y + \eta_{t-1} c_m + \eta_{t-2} c_r \leq \eta_t w_y + \eta_{t-1} w_m$$

where η_τ is the population size of the cohort born in $\tau = t, t-1, t-2$. As with the fixed population model, the competitive equilibrium for this model cannot be strongly stationary, and hence we again work with simulations of the recursive equilibrium. For the simulation, we use the same parametrization as for the model with constant population: $w_y^y = 3/12$, $w_m^m = 5/12$, $n = .8$, $N = 1.2$, and $\delta = \left(\eta_y^y \cdot w_y^y + \eta_m^m \cdot w_m^m \right)/2$.
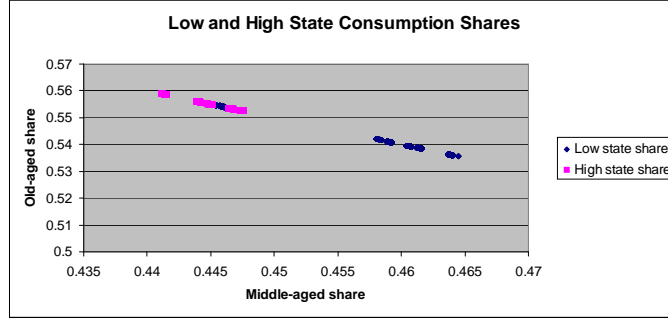
29

Figure 7: Some caption

The results for this version of the model parallel those of the fixed population case. Old agents bear a great deal of risk, and there exist opportunities for short-run Pareto improvements in risk-sharing between old and middle-aged agents. The table below shows the covariance matrix generated by the simulation between c_y^y, c_m^m, and c_r^r. From the variance numbers, the old bear more than 6 times the risk of the young for this version of the model.

0.019456	-0.000754	0.014966
-0.000754	**0.022779**	0.043581
0.014966	0.043581	**0.143634**

When we replicate the net-share analysis for this version of the model that we did for the fixed-population version, we get the following diagram on the share relationships.

This diagram is based on a simulated time-series of 1000 periods, and compares the transition from state 0 to either itself or state 4. As in the fixed-population case, we see the same general patten in which the old consume a higher share of net resources in the high state (state 4 in this case) than in the low state, which again indicates that there are risk-sharing improvements available. Note that there are some low state shares in the high state range in the data. This occurs because the transition from state 1 is to either state 0 or state 4. This transition has the same resource configuration as the state 0 transition, but weights the middle-aged more heavily because of the large cohort of old agents.

The social optimum for this version of the model is the solution to the

| Table 3: test |
State	EU at Social Optimum	EU at CE
0	-1.41740	-1.47286
1	-1.66273	-1.69497
2	-1.33518	-1.36426
3	-1.53218	-1.56536
4	-1.42902	-1.46994
5	-1.63986	-1.67929
6	-1.35431	-1.37652
7	-1.52844	-1.57316

problem

$$\max_{(c_y, c_m, c_r)} \eta_t Eu\left(c_y\right) + \eta_{t-1} Eu\left(c_m\right) + \eta_{t-2} Eu\left(c_r\right)$$

subject to

$$\eta_t c_y + \eta_{t-1} c_m + \eta_{t-2} c_r \leq \eta_t w_y + \eta_{t-1} w_m + D$$

where we weight each cohort by its size. We can also define this problem in *ex post* terms, if we wish. The solution is the same as in the fixed-population model, in the sense that at the optimum (and assuming the discount rate is zero) each agent consumes an equal share of the realized endowment. As before, we can also show that this is the optimal risk-sharing allocation. For our calibration of the model, the following table shows the expected utilities of an agent born in each state, given the possible transitions from that state.

Hence, as in the fixed-population model, the socially optimal allocation dominates the competitive allocation state by state, although in this case, the utility improvement is significantly less in percentage terms, averaging roughly 2.5%.

As with the fixed-population model, the second welfare theorem is applicable, though again with the caveat that it must be implemented by taxing away the dividend completely. So, as with the fixed-population model, we ran several experiments with taxes and transfers working in conjunction with competitive markets to see whether these would improve risk-sharing and increase welfare. Since the modelling set up for these experiments was similar to those for the fixed population model, we omit the details, except to note that for these experiments, we adopt the same parametrization as Geanakoplos et al. (2004). The diagram below shows the conditional ex-
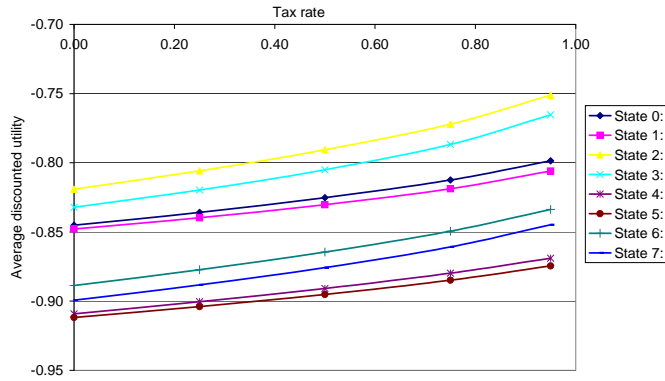
Figure 8: Low and high state consumption shares

State	Price
0	104.69
	(2.765)
1	91.52
	(1.531)
2	170.56
	(5.857)
3	143.47
	(4.075)
4	123.79
	(2.647)
5	111.09
	(1.964)
6	187.55
	(3.506)
7	157.95
	(3.910)

pected utility for agents born in each of the different population states for tax rates of 0%, 25%, 50%, 75% and 95%. As in the fixed population model, these expected utilities are monotonically increasing as we increase the tax rates.

We note finally that we also obtain the Geanakoplos et al. (2004) result on the predictability of asset prices. The following table shows that asset prices for each state together with their standard errors. The chart below shows the asset prices together with error bars going out two standard deviations.

From the chart, it is clear that states 0 and 5 have some overlap, as do 2 and 6, 2 and 7, and 6 and 7. However, none of these overlapping prices

Figure 9: Unconditionally expected utility at different tax rates

occur for possible state transitions, so that asset prices will be predictable from one state to the next.

7   Conclusion

The analysis we have presented demonstrates unambiguously that the *laissez-faire* competitive equilibrium in a multi-period OLG economy with productive assets will be Pareto suboptimal because of imperfect risk-sharing. The deviation from the first welfare theorem arises because of the restricted market participation imposed on unborn agents by the finite lifetimes assumption underlying the OLG environment, and the endogenous market-incompleteness generated by the weak stationarity of the competitive equilibrium in the multi-period setting.

We established that in a OLG economy where the exogenous state is governed by a Markov process, the strongly stationary *ex ante* optimal allocation is characterized by that the *cross sectional* marginal rates of substitution between two consecutive age groups are equal to 1. The fundamentally interesting result is that from any competitive equilibrium state, it is possible to construct a sequence which will make every individual *conditionally* weakly better off, at least one individual strictly better off, and which will end up in the *ex ante* strongly stationary optimal allocation in a finite num-

ber of periods.

On a very fundamental level, these results also have clear and obvious policy implications for the ongoing debate over whether governments should provide social insurance. Compared with a situation where the government has no role in redistributing income across generations, our exercise shows that government intervention can improve upon the risk sharing between the individuals and therefore the welfare of everybody. The exact extent of government intervention is harder to quantify. Since the two factors of production are supplied inelastically in our model economy, we are also ignoring any potential tax-induced distortions which might reduce welfare. In a world where governments can not solely rely on lump-sum taxation, there will exist trade-offs between risk sharing and efficiency in production. We leave this open for further research.

References

Abel, A. B., N. G. Mankiw, L. Summers and R. J. Zeckhauser (1989). "Assessing dynamic efficiency: Theory and evidence." *Review of Economic Studies*, **56**, 1–20.

Aiyagari, S. R. and D. Peled (1991). "Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models." *Journal of Economic Theory*, **54**(1), 69–83.

Ball, L. and N. G. Mankiw (2001). "Intergenerational risk sharing in the spirit of Arrow, Debreu, and Rawls, with applications to social security design." Tech. rep., Harvard.

Bohn, H. (2001). "Intergenerational risk sharing and fiscal policy." Tech. rep., UCSB.

Cass, D. (1972). "On capital overcumulation in the aggregate neoclassical model of economic growth: A complete characterization." *Journal of Economic Theory*, **4**, 200–223.

Chattopadhyay, S. and P. Gottardi (1999). "Stochastic OLG models, market structure, and optimality." *Journal of Economic Theory*, **89**, 21–67.

Demange, G. (2002). "On optimality in intergenerational risk sharing." *Economic Theory*, **20**, 1–27.

Gale, D. (1973). "Pure exchange equilibrium of dynamic economic models." *Journal of Economic Theory*, **5**, 12–36.

Geanakoplos, J., M. Magill and M. Quinzii (2004). "Demography and the long-run predictability of the stock market." *Brookings Papers on Economic Activity*, (1).

Kubler, F. and D. Krueger (2002). "Intergenerational risk sharing via social security when financial markets are incomplete." *American Economic Review.*, **92**(2), 407–410.

Monderer, D. and L. S. Shapley (1996). "Potential games." *Games and Economic Behavior*, **14**, 124–143.

Muench, T. (1977). "Optimality, the interaction of spot and futures markets, and the nonneutrality of money in the lucas model." *Journal of Economic Theory*, **15**, 325–44.

Peled, D. (1982). "Informational diversity over time and the optimality of monetary equilibria." *Journal of Economic Theory*, **28**, 255–74.

Smetters, K. (2004). "Trading with the unborn: A new perspective on capital income taxation." Tech. rep., UPenn.

Spear, S. E. (1985). "Rational expectations in the overlapping generations model." *Journal of Economic Theory*, **35**, 251–75.

Zilcha, I. (1990). "Dynamic efficiency in overlapping generations models with stochastic production." *Journal of Economic Theory*, **52**, 364–79.