Reputation and Impermanent Types

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Abstract

I consider a version of the chain store game where the incumbent firm's type evolves according to a Markov process with two states: a "tough" type who always fights entry, and a "weak" type who prefers to accommodate. There exists a minimal level of persistence necessary for the incumbent to be able to sustain any reputation for being tough. Above that level, as the number of markets *T* increases, in equilibrium play alternates between intervals of entry by competitors and intervals of deterrence. When *T* is infinite, then regardless of the discount factor there exists a sequential equilibrium in which the incumbent's payoff is bounded below her Stackelberg payoff. Both results are in contrast to the outcome when the incumbent's type is fixed. One interpretation is that reputation is not permanent, but must be renewed occasionally.

1. Introduction

In Selten's (1978) chain store game, a single long-lived incumbent firm (the "chain store") faces a sequence of *T* potential entrants ("competitors") in distinct, identical markets. Each competitor lives for only a single period. He chooses whether to enter the market and compete with the incumbent, or to stay out. Whether or not entry is profitable depends on the incumbent's subsequent decision to accommodate the entry or to fight it through a price war. In the short run, accommodation is the better response for the incumbent, but she may decide to fight if doing so will deter future competitors from

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entering. Selten (1978) shows, however, that when T is finite, there is a unique subgame perfect equilibrium, in which all competitors enter and the incumbent always accommodates. In this paper, I consider the case that the incumbent may be one of two types: the regular, or *weak*, type, who prefers accommodation to fighting, and the *tough* type, who prefers to fight even in the short run. The incumbent's type evolves following a Markov process. If the Markov process has at least a certain minimal level of persistence, then as the number of markets T increases, in equilibrium play alternates between intervals of deterrence (in which the competitor enters with probability zero, out of fear of being fought) and intervals of entry (with positive probability of entry). Price wars – where entry occurs and is fought – arise in equilibrium with a probability that approaches 1 as T grows. However, as the persistence of the Markov process also increases, entry is deterred in nearly 100% of the markets. When the number of markets is infinite, then for any parameterization of the game, there exists a sequential equilibrium in which the incumbent's payoff is bounded below her Stackelberg payoff. That bound is independent of the discount factor.

Selten's (1978) result is known as the chain store paradox: in the unique subgame perfect equilibrium of the finite-horizon game, no matter how many times the incumbent fights entry, the next competitor believes that she will accommodate if he enters. It is impossible for the incumbent to build a reputation for fighting, and thus she cannot deter entry. Selten (1978) models only the regular, or weak, type of incumbent. Kreps and Wilson (1982) and Milgrom and Roberts (1982) show that reputation can play a role in the chain store game by introducing a small probability that the incumbent is a tough, or crazy, type who always fights entry. In their models, types are permanent – a tough type remains tough for all T periods, and a weak type remains weak. Kreps and Wilson (1982) and Milgrom and Roberts (1982) show that in equilibrium, the competitor in the t-th-to-last market stays out if he believes that the incumbent is tough with probability greater than d^t, where $d \in (0, 1)$ is the probability of being fought that makes him indifferent between entering and staying out. That is, the threshold belief to deter entry falls exponentially away from the end of the game. One intuition is that near the last period, a reputation for toughness has little value, but when there are many markets remaining, the incumbent is more willing to fight. Early in the game, the chance that the incumbent will try to establish a reputation for being tough by fighting is itself enough to deter entry. In equilibrium, for any small *ex ante* probability p^0 that the incumbent is tough, entry is deterred in except in the last N periods, where N is the smallest integer such that $p^0 > d^N$. In the infinite horizon case, Fudenberg and Levine (1989) show that as the incumbent becomes more and more patient, the lower bound on her payoff in any Nash equilibrium approaches her Stackelberg payoff – that is, in equilibrium competitors sty out in nearly all markets.

In this paper, I again consider two types of incumbent, but, more realistically, allow her to switch between types. The tough incumbent may be one whose production costs are so low that she can easily win a price war with an entrant, and even benefit from the publicity generated by the competition. Under that interpretation, it is natural to suppose that the incumbent's type may change over time, as production costs vary due to, for example, fluctuations in the prices of inputs, or as consumer demand changes. Similarly, a firm facing a temporary cash crunch may be unwilling or unable to fight a price war.

Introducing impermanence into the incumbent's type qualitatively changes equilibrium play with a finite horizon. Not only can entry be deterred after accommodation (that is, the incumbent can regain a lost good reputation), but in fact deterrence can follow immediately after accommodation. As will be shown, in some cases the incumbent need not fight a price war to reestablish herself as tough. In the models of Kreps and Wilson (1982) and Milgrom and Roberts (1982), on the other hand, accommodation leads to entry in all subsequent markets. In the infinite horizon case, moving away from perfect persistence decreases the lower bound on the incumbent's payoff – in contrast to Fudenberg and Levine's (1989) result, there are equilibria where she gets (in expectation) strictly less than her Stackelberg payoff, regardless of the discount factor. With changeable types, a bad reputation does not last forever. The incumbent can thus accommodate occasionally without suffering permanent harm. That temptation to accommodate, however, may in turn lead more competitors to enter, so the overall effect is to lower the range of equilibrium payoffs for the incumbent.

Other research looking at games where types change over time includes the work of Tadelis (1999, 2002, 2003) and Mailath and Samuelson (2001), who study the market for firms' names (and thus their reputations) when such transactions are not observed by consumers. (A similar story provides another explanation for how an incumbent's type changes – a weak owner or manager might be placed by a tough one, without the public's knowledge. Aoyagi (1996) examines a version of the chain store game in which the owner of the firm changes, but not its type.) Holmstrom (1999) studies the behavior of a manager who wants to convince potential employers that his talent level, which follows a random walk, is high. Phelan (forthcoming) models capital taxation by a government that switches between being trustworthy and not. Similarly, in Cole, Dow, and English's (1995) model, governments alternate between being more or less myopic; the less myopic government is willing to repay sovereign debt. Athey and Bagwell (2004) study competition and collusion in procurement auctions between two suppliers, each of whom receives private and imperfectly persistent cost shocks. Finally, Cripps, Mailath, and Samuelson (2004, 2005) suggest a very different motivation for studying impermanent types. They demonstrate that in a simultaneous-move, infinite-horizon game with imperfect monitoring, a reputation cannot last forever: eventually the temptation to exploit it is too great. They conclude, then, that for reputation to be more than a transitory phenomenon in such a setting, it must be renewable – types must be changeable.[1]

In the current model, the incumbent's type switches between tough and weak according to a Markov process: a tough type becomes weak with probability q, and a weak type becomes tough with probability $r < 1 - q$. The stationary probability of a tough type, $r/(r + q)$, is assumed to be small, so that *ex ante* the incumbent is very likely to be the regular type, as is usual in the literature on reputation. Also, $1 - q > d$ and $r < d$, so that the incumbent's being known to be tough in one period suffices to deter entry in the next period, but the chance that a weak incumbent becomes tough does not. A couple of results follow immediately. If the number of markets T is fixed (and finite), then as the Markov process approaches perfect persistence (as q and r approach 0), then equilibrium payoffs approach those from Kreps and Wilson's (1982) and Milgrom and Roberts' (1982) permanent-type case. As r rises toward $1 - q$ (no persistence),

[1] Ekmekci (2005) argues that restrictions on the observability of past signals may also keep reputation from permanently vanishing.

equilibrium payoffs tend toward those of the i.i.d. case, where no reputation can be sustained.

The interesting case is when $r < d < 1 - q$ are fixed, and the number of markets T increases. With permanent types, the finite-horizon equilibrium features a long period (in fact, increasing without bound as T grows) of deterrence at the beginning of the game, followed by entry at the end. With changing types, in contrast, play alternates between stretches of deterrence and periods where entry has positive *ex ante* probability. Roughly, the intuition is as follows: in Kreps and Wilson's (1982) and Milgrom and Roberts' (1982) permanent-type model, the cutoff probability of the incumbent's being tough, d_t, that deters entry in the t-th-to-last market falls exponentially with t. A similar process emerges with Markov types. Eventually, though, d_t falls below r, the probability that the incumbent is tough today when she was known to be weak yesterday. But then a weak incumbent has no incentive to fight in the (t + 1)th-to-last market – entry will be deterred tomorrow even if she accommodates and reveals herself as weak. Thus, only a tough incumbent will fight, and the cutoff deterrent probability d_{t+1} jumps back up to d. In fact, as T increases, the probability that in equilibrium entry occurs and is fought (a "price war") approaches one. Nevertheless, it can be shown that the limiting (for large T) fraction of markets in which entry is deterred tends toward one as the persistence of the incumbent's type grows.

When the number of markets T is infinite, then the finite-horizon strategies can be modified to construct a sequential equilibrium that cycles between entry and deterrence. The expected payoff to the incumbent in that equilibrium is strictly less than her Stackelberg payoff, and it does not vary with her discount factor. As a corollary, when the incumbent is patient enough, that cycling equilibrium can be used as a threat to support an equilibrium in which all competitors stay out, and the incumbent would always fight entry.

The tough type of incumbent will be modeled in two ways, either as a committed type who must fight whenever given the chance, or as a strategic type who has the same action set as the weak type, but who prefers fighting to accommodation conditional on entry, and no entry to fighting. It turns out that, subject to a restriction on beliefs, both specifications yield the same outcome with a finite horizon. That is, the strategic type will always choose to fight in equilibrium. The relevant restriction on beliefs is that after observing a choice of "fight," competitors cannot revise their belief that the incumbent is tough downward. Tough types are at least as likely to fight as weak types, even off the equilibrium path. The equilibria constructed in the infinite horizon version also are valid for either specification of the tough type.

In the next section I describe the model. Section 3 contains a simple example. The results with finite and infinite horizons are in Sections 4 and 5, respectively. Section 5 is the conclusion.

2. Model

There are $T \in [1, \ldots, \infty]$ periods and $T + 1$ players: a single long-run incumbent, active in every period, and a sequence of T short-run competitors, each active in a single period only. For the finite horizon case, let period t denote the t-th-to-last period, and player t the competitor active in that period. (That is, time is counted backward from the end of the game.) In the infinite horizon case, time is counted forward as usual.

In any period, the incumbent may be either a weak type or a tough type. Suppose that in period t she is weak. At the start of the period, competitor t chooses whether to *Enter* market t or to *Stay Out*. If the competitor stays out, then he gets a payoff of 0, and the incumbent gets payoff $a > 1$. If the competitor enters, then the incumbent must choose whether to *Fight* the entry or to *Accommodate* it. If she fights, then both players get a payoff of –1. If she accommodates, then she gets a payoff of 0, and the entrant gets $b > 0$. That extensive form stage game is illustrated in Figure 1. A competitor's total payoff from the T-period game is just his payoff from the stage game in the single period in which he is active. Thus, the competitors have no strategic motivation. Each wants only to maximize his payoff in the stage game. They will enter, then, whenever the probability of being fought is less than $d \equiv b/(b + 1)$.

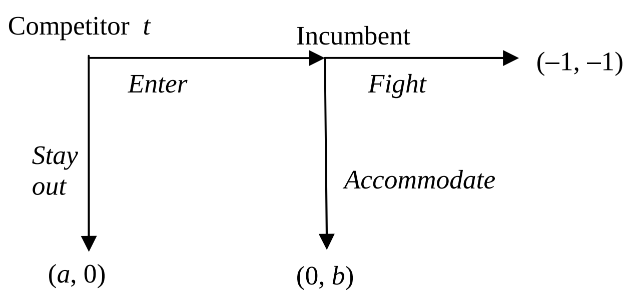

Figure 1: Stage-game Payoffs with a Weak Incumbent

The payoffs and action sets for the incumbent given above apply when she is a weak type. The other type of incumbent is the tough type, which will be modeled in two ways. The first way is to suppose that a tough incumbent has no choice but to fight whenever the competitor enters. Her payoffs from the stage game are the same as the weak type's. That is, the tough incumbent is a committed type. The other way is to model her as a strategic type, who has the same set of available actions in the stage game as the weak type, but who prefers fighting to accommodation. For the sake of simplicity, the first way of modeling will be used in the rest of the body of the paper. In the Appendix, it will be shown that the second modeling choice yields the same equilibrium outcome as the first, subject to a restriction on the competitors' off-equilibrium beliefs: they cannot revise their belief that the incumbent is tough downward after seeing her choose to fight rather than accommodate. That equivalence holds as long as the strategic tough type prefers fighting to accommodating, conditional on entry, and prefers the competitor's staying out to fighting entry, by the same margin a that the weak incumbent prefers no entry to accommodating. The fact that the tough incumbent prefers no entry rules out equilibria where a tough incumbent might accommodate entry in order to induce future competitors to enter as well, and the fact that the margin of preference is a implies that a weak incumbent's incentives to deter entry are unchanged relative to the equilibrium with a commitment type.

With either way of modeling the tough type, the incumbent's total payoff is the sum of her stage game payoffs in the T markets. In the infinite horizon case, those payoffs are discounted at a rate δ per period, and their sum is weighted by $(1 - \delta)$. With a finite horizon, for ease of exposition the incumbent is assumed to be perfectly patient. That simplification does not affect the results (except that if she is very impatient, she will have no incentive to build a reputation – the future gain from deterring entry is outweighed by the current cost of fighting).

The incumbent's type evolves period by period according to a Markov process. A tough type becomes weak with probability $q > 0$, and a weak type becomes tough with probability $r > 0$, where $r < 1 - q$. (The case of $q = r = 0$ corresponds to perfect persistence of the Markov process. When $r = 1 - q$, there is no persistence.) The stationary distribution of the process is a probability of being tough $p^0 \equiv r/(r + q)$. Let p^0 also be the initial probability that the incumbent is tough in the first market in period T. That simplification economizes on notation, but is not necessary for the results. The parameters of the game are common knowledge, and the outcomes of previous stage games are observed by all players. (That is, the incumbent's action is observed only if the competitor enters.) The incumbent's type, however, is known only to her. She privately observes her type at the beginning of each period.

In keeping with the usual assumption in the literature on reputation that the tough type occurs with only small likelihood, let the limiting probability p^0 be less than the cutoff deterrent probability d. Also, let d be less than $1 - q$, the probability that a tough type remains tough. If d were greater than $1 - q$, then in the last period, when only a tough incumbent will fight (a weak one has no future reputation to protect), the competitor will enter for sure. Even if the incumbent was known to be tough in the previous period, the probability that she is tough today is only $1 - q$, which is too low to deter entry. Backwards induction, then, yields the same unraveling as in Selten (1978). In the unique sequential equilibrium when $d > 1 - q$, every competitor enters, and the incumbent accommodates whenever she is weak. Thus, a tough type must remain tough with a probability at least d for any reputation to be sustainable, and so let $0 < r < r/(r + q) < d < 1 - q$.

A pure (behavior) strategy maps public histories to a choice of entering or staying out for competitors, and maps complete histories (including the history of privately observed types) to a choice of fighting or accommodating for the weak incumbent. It will be useful to introduce the function M: $[0, 1] \rightarrow [r, 1 - q]$, defined as $M(p) = (1 - q)p + r(1 - p)$, which gives the probability that the incumbent will be tough tomorrow given that she is tough today with probability p. Let the function W: $[r, 1 - q] \rightarrow [0, 1]$ be the inverse of M. If tomorrow's probability of being tough is to be p, then today's must be $W(p) = \frac{p - r}{1 - q - r}$. Also, let p_t denote the probability at the beginning of period t that the competitors assign to the incumbent's being tough in that period. That probability is based on the public history of actions, the parameters of the Markov process, and the incumbent's strategy.

3. Examples

This section contains two partially parameterized examples to illustrate the dynamics of reputation when types are changeable. The first example looks at a three-period game, and the second extends the first to a large number of markets.

Example 1: Suppose that the number of markets T is 3 and that the cutoff probability of being fought to deter entry d is 0.53. Suppose also that both the probability that a weak incumbent becomes tough r and the probability that a tough one becomes weak q are equal to 0.4. The stationary and initial probability of being tough p^0, then, is 0.5. The sequential equilibrium of this game can be solved through backwards induction. Since the tough incumbent is a committed type who always fights, only the strategies of the competitors and the weak incumbent, as well as the competitors' beliefs, need to be calculated.

In the last market, in period 1, the weak incumbent's best response to entry is to accommodate. The competitor, then, will enter if his belief that the incumbent is tough p_1 is less than $d = 0.53$.

In period 2, the weak incumbent will fight if the competitors' current belief p_2 that she is tough is greater than $W(0.53) = 0.65$. In that case, competitor 1 gets no information about her type from her action, and so his belief in period 1 will be $M(p_2) > 0.53$; he will stay out. On the other hand, if the weak incumbent reveals herself as weak by accommodating in period 2, then competitor 1's belief will be $M(0) = r = 0.4$, and he will enter. Since $a > 1$, it is worthwhile for the incumbent to fight in period 2 in order to deter entry in period 1: $-1 + a > 0 + 0$.

If the competitors' belief p_2 is below 0.65, then in equilibrium the weak incumbent must play a mixed strategy, just as in Kreps and Wilson (1982) and Milgrom and Roberts (1982). If she fights with probability one, then competitor 1's belief p_1 will be $M(p_2) < 0.53$, and he will enter. Since fighting does not deter entry in that case, the weak incumbent would prefer to accommodate. But if the weak incumbent accommodates with probability one, competitor 1 will conclude after observing the incumbent fight that she is tough for sure. His belief p_1 will be $M(1) = 0.6 > 0.53$, and he will stay out. Now the weak incumbent would prefer to fight, since doing so would deter entry in the last market. Thus, equilibrium must involve mixing. The weak incumbent is willing to mix if competitor 1 enters with probability $(a - 1)/a \in (0, 1)$ after observing the incumbent fight in period 2; he enters for sure if he sees accommodation. Competitor 1, in turn, is willing to randomize only if his belief p_1 is equal to the cutoff probability d. If the weak incumbent chooses to fight with probability $\beta(p_2) \equiv \frac{p_2}{1-p_2}\frac{1-W(d)}{W(d)}$, then the competitors' updated belief, after observing her fight in period 2, that she was tough in that period is $W(d)$. Competitor 1's belief p_1, then, will be $M(W(d)) = d$, as desired.

Thus, in equilibrium the weak incumbent will choose to fight in market 2 with probability $\beta(p_2)$ when $p_2 < 0.65$, and with probability one if $p_2 > 0.65$. The total probability that entry will be fought, then, is the smaller of $p_2 + (1 - p_2)\beta(p_2) = p_2/0.65$ and 1. Competitor 2's best response is to enter if $p_2 < dW(d) = (0.53)(0.65) = 0.3445$. Since $M(0) = 0.4$, though, even if the incumbent in period 3 revealed herself as weak, competitor 2's belief will be greater than his cutoff 0.3445. Therefore, he never enters.

In period 3, the weak incumbent will accommodate entry. By doing so, she reveals herself as weak, but competitor 2 will stay out anyway. Since competitor 2 stays out, no information is gained in period 2 about the incumbent's type, so competitor 1's belief p_1 will be $M^2(0) = 0.48 < 0.53$, and thus he will enter. If competitor 3 enters, the weak incumbent gets a total payoff of $0 + a + 0 = a$ from accommodating. She cannot do better by fighting. Competitor 2 will stay out in any case. Even if fighting in market 3 convinces the competitors that she is tough for sure, by period 1 their belief p_1 will have declined to $M^2(1) = 0.52 < 0.53$, which is too low to deter entry. Thus, fighting yields a payoff of $-1 + a + 0 < a$, and so the weak incumbent will accommodate. As in period 1, then, competitor 3 knows that only a tough incumbent will fight, and his best response is to enter if his belief $p_3 < d = 0.53$. Since the initial belief p^0 is 0.5, competitor 3 will enter.

In the unique sequential equilibrium, then, competitors 1 and 3 enter, and competitor 2 stays out. The weak incumbent accommodates the two entries, and the tough incumbent fights, as always.

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In Example 1, play switches from entry in the first market, to deterrence in the second, and then back to entry in the last market. Example 2 shows that the pattern of alternating entry and deterrence recurs as the number of markets increases.

<u>Example 2:</u> As in Example 1, let the cutoff probability of being fought to deter entry d be 0.53, and let the transition probabilities q and r both be 0.4. Now, though, suppose that the number of markets T is large. Equilibrium play in the last three periods is the same as in Example 1. Note that the minimum probability of facing a tough incumbent that deters competitor 3 from entering, 0.53, is the same as competitor 1's cutoff: both know that only a tough competitor will fight them. A weak incumbent deciding whether or not to fight in period 4, then, faces the same tradeoff as she does in period 2. In equilibrium, she must mix with probability $\beta(p_4)$ on fighting if $p_4 < 0.65$, and to fight for sure otherwise. (Similarly, in equilibrium competitor 3 must mix with the same probabilities as competitor 1 when he is indifferent, at $p_3 = d$.) Competitor 4's cutoff belief for entry, then, is 0.3445, the same as competitor 2's, and so he never enters.

Likewise, a weak incumbent in period 5 and competitor 5 behave just as their periods 1 and 3 counterparts do. A weak incumbent in period 6 thus finds herself in the same position as in periods 2 and 4, and so on.

In equilibrium, the weak incumbent accommodates in odd-numbered periods. In even-numbered periods, she fights with probability 1 when the competitor's belief p_t is at least 0.65, and with probability $\beta(p_t)$ otherwise. Competitors in odd-numbered periods enter when $p_t < d$, stay out when $p_t > d$, and randomize with probability $(a - 1)/a$ on entry when $p_t = d$. Competitors in even-numbered periods stay out. That equilibrium is unique. On the equilibrium path, play cycles between entry, met by accommodation from weak incumbents and a fight from tough ones, in odd periods, and deterrence in even periods, regardless of the number of markets T. That pattern is very different from the outcome in the permanent-type case, in which entry can occur only in the last few markets, and thus the fraction of markets in which entry is deterred approaches 1 as T grows.

4. Results – Finite Horizon

The situation described in Example 2 is atypical in two ways. First, in the odd-numbered periods, the incumbent deters entry for exactly one period either by accommodating (and revealing herself as weak), or by fighting, even if fighting convinces competitors that she is tough for sure. The reason is that there is very little persistence in the incumbent's type, so that the probability of being tough two periods after being weak (0.48) is nearly the same as it is two periods after being tough (0.52). Both probabilities are below the cutoff for deterrence, 0.53. More generally, though, if the Markov process has greater persistence, beliefs are slower to converge from the extremes to the stationary probability p^0. Thus, the fact that being revealed as weak today still deters entry tomorrow does not imply that a weak incumbent will accommodate: being revealed as tough may deter entry for more than one period. The basic intuition of the example, however, is robust. As will be shown in Theorem 1, given any parameters of the game, eventually even being revealed as weak is enough to deter entry for a long enough time that beliefs have a chance to converge very close to p^0, regardless of where they started. That result will imply that play alternates being entry and deterrence.

The second way in which Example 2 is unusual is that there is no randomization on the equilibrium path. Competitors either strictly prefer to enter or strictly prefer to stay out, and in periods when they enter a weak incumbent always accommodates. In general, there may be non-trivial mixing in equilibrium, which makes the distinction between "periods of entry" and "periods of deterrence" less straightforward, for two reasons. Besides the fact that a competitor may choose to enter or stay out randomly, beliefs in a period (and thus the competitor's action) may depend on the outcomes of prior randomizations. For the sake of uniformity, define a "period of entry" as one in which the *ex ante* probability that the competitor enters is strictly positive. That is, period t is a period of entry if there is a path of play that occurs in equilibrium with positive probability along which the competitor enters in market t. Otherwise, t is a "period of deterrence."

In order to establish the results of Theorems 1 through 3 (that play alternates between entry and deterrence, that price wars arise in equilibrium, and that entry is nearly always deterred when the incumbent's type is very persistent), the sequential equilibrium of the game, given the number of markets T, payoffs a and b, and transition probabilities q and r, is constructed below. The derivation will show that i) the equilibrium is generically unique, ii) behavior strategies depend only on the period and the competitors' current (public) beliefs and not otherwise on history, and iii) in each period there is a cutoff probability that the incumbent is tough such that the competitor will enter if beliefs are below the cutoff and stay out if they are above. Remember that d ($\equiv b/(b + 1)$) is the minimal probability of being fought that deters entry, p^0 ($\equiv r/(r + q)$) is both the initial and stationary probability that the incumbent is tough, and p_t is the competitors' belief at the start of period t. In addition, define d_t as the cutoff belief to deter entry in period t. Note that d_t cannot be greater than d, which deters entry even when only a tough incumbent fights. Let μ_t be the probability that the period-t competitor enters when he is indifferent (at $p_t = d_t$). If $d_t < r$, then μ_t does not exist, because p_t cannot be less than r. Also, μ_T typically does not exist, since generically the initial belief p^0 does not equal d_T. Let the function $\beta_t(p_t)$ give the probability that a weak incumbent fights in period t, conditional on entry and p_t.

The equilibrium strategies are defined recursively below: d_t and $\beta_t(\cdot)$ are given in Step t, and μ_t is defined in Step $t + 1$. Beliefs come from Bayes' rule on the equilibrium path (that is, at histories reached with positive probability in equilibrium). There are two ways for a deviation to lead off the path. If a competitor who is supposed to stay out enters, then beliefs are updated according to the incumbent's continuation strategy and the Markov parameters, just as in equilibrium. If an incumbent who is supposed to fight with probability one in period t accommodates, she is believed to be weak for sure, and so $p_{t-1} = M(0) = r$. (Conditional on entry, fighting always occurs with positive probability in equilibrium, because the incumbent may be tough.)

Step 1: As in the examples, a weak incumbent will always accommodate, so $\beta_1(p) = 0$ for all p. Consequently, $d_1 = d$.

Step 2: Again as in the examples, a weak incumbent will fight for sure if $p_2 > W(d_1)$, since doing so will deter entry in period 1. If $p_2 \le W(d_1)$, then $\beta_2(p_2) = \dfrac{p_2}{1-p_2}\dfrac{1-W(d_1)}{W(d_1)}$, so that if competitor 1 sees the incumbent fight in market 2, his belief p_1 will equal d_1, and thus he will be indifferent between his actions. To make the weak incumbent willing to mix in period 2, it must be that $\mu_1 = (a - 1)/a$. Given $\beta_2(\cdot)$, the total probability of being fought is $\min\left\{\dfrac{p_2}{W(d_1)},1\right\}$, so the cutoff for entry d_2 is $dW(d_1)$.

Step t: Define $n(t)$ as follows:

$$n(t) = \min\left\{n \in \{1,\ldots,t-1\} : M^n(0) \le d_{t-n}\right\},$$

and define $s(t)$ as $t - n(t)$. (Note that the constraint set must be nonempty, because $M^{t-1}(0) < p^0 < d = d_1$.) The value $n(t)$ is the number of periods that the Markov process must act, starting from probability 0 at time t, until the expected probability of being tough is below the cutoff for deterrence. That is, if the incumbent is known to be weak in period t, and no further information about her type is revealed for $n(t)$ periods, then in period $s(t)$ the competitor will enter, and not before. (Generically, the case that $M^{n(t)}(0) = d_{s(t)}$ does not arise, and so from here on it is ignored.) To illustrate the definition, in Examples 1 and 2, $n(2) = 1$, since $M(0) < d_1$, and $n(3) = 2$, since $d_2 < M(0)$ and $d_1 > M^2(0)$.

There are two cases, according to whether or not being known to be tough in period t deters entry in market $s(t)$; that is, whether $M^{n(t)}(1) > d_{s(t)}$ or $M^{n(t)}(1) < d_{s(t)}$. (Again, generically the case of equality does not arise.) Note that for $n < n(t)$, the definition of $n(t)$ implies that $M^n(1) > d_{t-n}$, since $M^n(1) > M^n(0)$. In the first case, where $M^{n(t)}(0) < d_{s(t)} < M^{n(t)}(1)$, period $s(t)$ is the earliest period in which the incumbent's behavior in period t can affect his future payoff. In previous periods, the competitor will stay out regardless of what he observed in period t. If $p_t \le W^{n(t)}(d_{s(t)})$, then the weak incumbent must randomize, just as in period 2 – always fighting will not deter entry in period $s(t)$, but if she never fights, then a deviation to fighting will convince the competitors that she is tough for sure, which will prevent competitor $s(t)$ from entering. The incumbent fights with probability $\beta_t(p_t) = \dfrac{p_t}{1-pt}\dfrac{1-W^{n(t)}(d_{s(t)})}{W^{n(t)}(d_{s(t)})}$, so that competitor $s(t)$ is indifferent between his two actions. For the incumbent to be willing to mix requires that $\mu_{s(t)} = (a-1)/a$, which pins down the value of μ_{t-1} if $n(t) = 1$. If $n(t) > 1$, then $\mu_{s(t)}$ was defined in an earlier step. Note, though, that in any period x where $d_x > r$, $s(x+1) = x$, since $M(0) = r < d_x \le d < 1 - q = M(1)$, and so $\mu_x = (a-1)/a$. Since $d_{s(t)} > M^{n(t)}(0) \ge r$, then, $\mu_{s(t)} = (a-1)/a$, as required. If $p_t > W^{n(t)}(d_{s(t)})$, then the weak incumbent will fight for sure, since doing so suffices to deter entry in period $s(t)$.

The total probability that entry will be fought, then, is $\min\left\{\frac{p_t}{W^{n(t)}(d_{s(t)})},1\right\}$, so the cutoff for entry d_t is $dW^{n(t)}(d_{s(t)})$.

The second case is $M^{n(t)}(0) < M^{n(t)}(1) < d_{s(t)}$. Again, before market $s(t)$ all competitors will stay out, regardless of what happens in market t, but in this case competitor $s(t)$ will enter even if the incumbent was known to be tough in period t. In period t, then, a weak incumbent will always accommodate, since her continuation payoff from $s(t)$ on does not vary with that period's belief $p_{s(t)}$ for $p_{s(t)} < d_{s(t)}$: the weak incumbent in that case is willing to mix in period $s(t)$, and so gets the payoff from accommodating and revealing herself as weak regardless of $p_{s(t)}$. That is, the incumbent's action in period t affects beliefs but not payoffs in period $s(t)$. The weak incumbent's probability of fighting in period t $\beta_t(p_t)$, then, is zero, so $d_t = d$.

To summarize, if $M^{n(t)}(1) > d_{s(t)}$, then $\beta_t(p_t) = \min\left\{1, \frac{p_t}{1-pt}\frac{1-W^{n(t)}(d_{s(t)})}{W^{n(t)}(d_{s(t)})}\right\}$ and $d_t = dW^{n(t)}(d_{s(t)})$. If $M^{n(t)}(1) < d_{s(t)}$, then $\beta_t(p_t) = 0$ and $d_t = d$. The value of μ_{t-1}, if it exists, is $(a-1)/a$.

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The recursive formulation above defines the equilibrium, but it does not immediately provide much insight into its characteristics. Theorem 1 below demonstrates that if the number of markets T is large enough, then the equilibrium features alternating stretches of periods of deterrence and periods of entry. The method of proof is first to show that starting in any market t^0 and working backwards away from the end of the game, there must eventually be a period of deterrence, because the cutoff probability d_t falls below r. The intuition for that result is similar to Kreps and Wilson's (1982) and Milgrom and Roberts' (1982). Second, again working backwards from any t^0, there must eventually be a period of entry. Otherwise, roughly, there must at some point occur a long string of very low d_t's, so that in the period s immediately prior even accommodating will deter entry for the duration of the string. If the string is long enough, then (since no new information is revealed when no competitor enters) beliefs at the end are nearly independent of the starting belief. In that case, the weak incumbent has no incentive to fight in period s and will accommodate, as in period 3 in the examples. That result turns out to imply that entry must occur. To summarize, then, the proof of Theorem 1 shows that before any t^0, there must be a period of deterrence and also a period of entry, as long as T is large enough. That is, entry and deterrence must alternate.

Theorem 1: Let t^0 be given. For generic values of the parameters a, b, q, and r there is a $T^0 < \infty$ such that whenever $T \geq T^0$, the unique sequential equilibrium has the following property: there exist t^d, $t^e \geq t^0$ such that period t^d is a period of deterrence and period t^e is a period of entry.

Proof: The generic uniqueness is derived in the recursive formulation of the equilibrium above. Next, I show that there exists $t^d \geq t^0$ such that period t^d is a period of deterrence, as long as $T \geq t^d$. It suffices to show that for large enough t, the cutoff probability for deterrence d_t falls below r, since beliefs can be no lower than r. Note that if $d_t > r$, then $d_{t+1} = dW(d_t)$, as shown in the derivation of the equilibrium. Since $d_t \leq d < 1 - q$, for all $d_t > r$ the ratio

$$\frac{d_{t+1}}{d_t} = \frac{dW(d_t)}{d_t} = \frac{d(d_t - r)}{d_t(1-q-r)} < \frac{(1-q)(d_t - r)}{d_t(1-q-r)} = \frac{d_t(1-q)-(1-q)r}{d_t(1-q)-d_t r} < 1,$$

and the partial derivative of the ratio with respect to d_t

$$\frac{\partial\left(\frac{d_{t+1}}{d_t}\right)}{\partial d_t} = \frac{dr}{(d_t)^2(1-q-r)} > 0.$$

Thus, as t increases the ratio $d_{t+1} / d_t < 1$ shrinks at a increasing rate as long as $d_t > r$, and so eventually d_t must fall below r.

It remains only to prove the existence of t^e. It suffices to show that $d_s > p^0$ for some $s \geq t^0$: if there is no entry in the n periods before s, where n is large enough that $M^n(1) < d_s$, then the period- s competitor must have belief $p_s < d_s$ and will therefore enter, regardless of the belief at period $s + n$. Suppose that there is no such s. Then there must be a period $x < t^0$ such that $d_{x\text{-}1} > p^0$ and $d_t < p^0$ for all $t \geq x$. (Once again, the nongeneric case of equality is ignored.) Let $n(1)$ be chosen large enough that $M^{n(1)}(1) < d_x$, and let $d(1) = \max\{d_t : x \leq t \leq t^0 + n(1)\}$. In period $t^0 + n(1)$, then, $d_{t^0 + n(1)} \leq dW(d(1))$: belief $d(1)$ is sufficient to deter entry up to period x, and no belief is sufficient to deter entry longer. Since $d(1) < p^0$ by assumption, $dW(d(1)) < d(1)$, and so in fact $d_t \leq dW(d(1))$ for all $t \geq t^0 + n(1)$. Define $d(2)$ to be equal to $dW(d(1))$. By similar reasoning, then, if $n(2)$ is large enough that $M^{n(2)}(0) > d(1)$, then $d_t \leq d(3) \equiv dW(d(2))$ for all $t \geq t^0 + n(1) + n(2)$. No belief can deter entry past period x, and belief $d(2)$ deters entry up to period $t^0 + n(1)$, by which point beliefs will have grown through the Markov process to at least $d(1)$, which deters entry up to period x.

Continuing in that fashion, eventually there is a period t^* such that for $t \geq t^*$, $d_t < r$ (since the ratio $d(j + 1) / d(j)$ shrinks to zero, as shown above). But then, since $M^1(0) = r$, at period $t^* + 1$ a belief of 0 will deter entry up to period x, and no belief can deter entry longer. Thus, in period $t^* + 1$ the weak incumbent's best response is to accommodate, and so $d_{t^*+1} = d > p^0$, in contradiction of the assumption that $d_t < p^0$ for all $t \geq x$. *Q.E.D.*

The result that the minimal deterrent probability d_t can drop below r implies that deterrence can immediately follow accommodation. That is, after the incumbent loses her reputation for toughness by accommodating, she may not need to be seen to fight to restore it. The threat that she would fight if given the opportunity may suffice to deter entry.

Theorem 2 shows that as the number of markets T increases, the probability that a price war (in which the competitor enters and the incumbent fights) occurs in equilibrium approaches one. In fact, for any $N > 0$, the probability that at least N such price wars occur approaches one.

Theorem 2: Let $N > 0$ and $\varepsilon > 0$ be given. For generic values of the parameters *a*, *b*, *q*, and *r* there is a $T^0 < \infty$ such that whenever $T \geq T^0$, the unique sequential equilibrium has the following property: the probability that action profile (*Enter*, *Fight*) will be played at least *N* times is at least $1 - \varepsilon$.

Proof: First, note that in every period in which the competitor enters, the probability that the incumbent fights is at least *r* (the minimum probability that she is tough).

The proof of Theorem 1 shows that as *T* increases, the number of periods *t* in which the cutoff belief for deterrence $d_t > p^0$ grows without bound. For any such *t*, choose $n(t)$ such that $M^{n(t)}(1) < d_t$. If there is no price war in the $n(t)$ periods before period *t*, then in period *t* the competitor's belief p_t must be below d_t – either there has been no entry in the previous $n(t)$ periods, in which case the inequality $M^{n(t)}(1) < d_t$ guarantees that the Markov process has driven beliefs down far enough, regardless of their starting point; or in any of the $n(t)$ periods where the competitor entered, the incumbent accommodated, in which case beliefs are even lower. If there is no price war in the $n(t)$ periods before period *t*, then, the period-*t* competitor will enter for certain.

Thus, for any *t* such that $d_t > p^0$, there must be entry in one of the $n(t) + 1$ periods ending in period *t*. Since the probability that an entry will be fought is at least *r*, and the number of such periods *t* grows without bound, the probability that at least *N* price wars will occur as *T* increases approaches one. *Q.E.D.*

The third and final main result for the game with a finite horizon concerns the frequency of entry as the persistence of the incumbent's type grows. With perfect persistence, Kreps and Wilson (1982) and Milgrom and Roberts (1982) show that that frequency converges to zero as the number of markets T increases. The same is true in the limit here. Theorem 3 shows that the frequency of entry that results when T increases to infinity, given the level of persistence, shrinks to zero as that level of persistence approaches one (holding the stationary probability fixed). An immediate consequence is that the incumbent's average payoff per period approaches a, her Stackelberg utility.

Theorem 3: Let $p^0 \in (0, 1)$ and be given. Let $\{(q_k, r_k)\}_{k=1}^{\infty}$ be a sequence of transition probabilities in $(0, 1)^2$ such that

i) $\lim_{k\to\infty}(q_k, r_k) = (0,0)$,

ii) $r_k < 1 - q_k$ for all k, and

iii) $r_k /(r_k + q_k) = p^0$ for all k.

Then for generic values of the parameters a and b such that $p^0 < d \equiv b/(b + 1)$, the unique sequential equilibrium has the following property: $\lim_{k\to\infty}\left[\lim_{T\to\infty}\sigma(a,b,k,T)\right] = 0$, where $\sigma(a,b,k,T)$ is the fraction of periods that are periods of entry, as a function of a, b, q_k, r_k, and T.

Proof: As shown in the proof of Theorem 1, as the number of markets T increases, the number of periods t such that $d_t > p^0$ and $d_{t-1} < p^0$ grows without bound. Recall from the construction of the equilibrium that the "target belief" in a period (that is, the posterior probability that the incumbent is tough after the competitor enters and the incumbent fights) is equal to $d_t / d > d_t > p^0$. Thus, by fighting entry in period t, the incumbent will deter entry at least until the next period $s < t$ when $d_s > p^0$ – the Markov process cannot drive a belief that starts above p^0 to a level below p^0.

In the equilibrium constructed, in any period the difference in the number of periods of deterrence following accommodating and the number of periods of deterrence after achieving the target belief (by fighting) is at most one period. Thus, even if the posterior belief at the end of period t is 0, entry will be deterred at least until period $s + 1$ (that is, $M^n(0) > d_{t-n}$ for $1 \le n \le t - s - 1$), and so periods $s + 1$ to $t - 1$ are periods of deterrence if the number of markets T is at least t. How long is that stretch of deterrence? Since $d_s > p^0$, and the equilibrium construction establishes that $d_{s+1} = dW(d_s)$, it must be that $d_{s+1} > dW(p^0) = dp^0$, which exceeds r_k as r_k shrinks to zero. Since $d_{s+1} < M^{t-s-1}(0)$, and the increasing function M converges to the identity function as r_k and q_k shrink to zero, the length of the stretch of deterrence $t - s - 1$ grows without bound as k increases. That is, the number of consecutive periods where the cutoff belief is below p^0 becomes very large, and all but the first of those periods must be a period of deterrence. (If T is between s and t, then since the initial probability that the incumbent is tough is p^0, all the periods up through period $s + 1$ are again periods of deterrence.)

On the other hand, the number of consecutive periods in which the cutoff belief is above p^0 (the only ones in which entry may occur) is bounded above as persistence grows. The highest a cutoff belief d_t can be is d. Starting from that level, how many periods n does it take until $d_{t+n} < p^0$? As shown in the derivation of the equilibrium, whenever $d_s > r$, $d_{s+1} = dW(d_s)$. As k grows, then, the function W approaches the identity function, and n approaches the smallest integer satisfying $d^n < p^0$.

Thus, the limiting (as T grows) fraction of the periods in which entry is deterred approaches one as k (and the persistence of the incumbent's type) increases. *Q.E.D.*

5. Results – Infinite Horizon

In the equilibrium of the game with a finite horizon, in each market strategies depend only on the number of periods remaining until the next time that the weak incumbent would accommodate entry with probability one. In the infinite horizon, it is possible to construct equilibria where such times re-occur at regular intervals. In the last several periods of each interval, play is the same as at the end of the finite-horizon equilibrium. At the beginning of the interval (that is, immediately after the period in which the weak incumbent accommodates for sure), there is added a stretch of periods in which no competitor enters. If that stretch is long enough, then beliefs at the end of it are nearly independent of the belief at the beginning, and so in the prior period the weak incumbent has nothing to lose by accommodating and revealing her type (just as in the second case of Step t in the construction of the finite-horizon equilibrium).

The intuition for why entry is deterred in the "extension periods" is as follows: if a competitor deviates by entering in one of those periods, then play immediately jumps ahead to later in the interval, to a period from the finite-horizon equilibrium in which the incumbent fights with high probability. That is, entry brings close the end of the interval, thus giving the incumbent the proper incentive to fight and deter entry. In this cycling equilibrium, which is described formally in Theorem 4 below, entry occurs infinitely often, and thus the incumbent expected payoff is strictly below her Stackelberg payoff a. (Recall that in this section time is counted forwards again, and that the incumbent discounts the future at a rate δ per period.)

Theorem 4: For generic values of the parameters a, b, q, and r, there exists an $\varepsilon > 0$ such that for any discount factor $\delta \in (0, 1)$, there is a sequential equilibrium in which the expected payoff to the incumbent is no greater than $a - \varepsilon$.

Proof: The proof is constructive. If $\delta < 1/a$, it is easy to verify that there is an equilibrium in which the weak incumbent accommodates in every period, since the cost (-1) of fighting rather than accommodating outweighs the discounted gain from deterring entry in the next period (δa). Competitors enter when their belief is below d. That equilibrium gives the incumbent a payoff strictly below a.

Suppose that $\delta > 1/a$. In the equilibrium to be constructed, in every period play is in one of $S^{**} + N$ stages, where the values of $S^{**} \geq 2$ and $N \geq 0$ are given below. Play begins in stage $S^{**} + N$. The transition rule between stages is as follows: if play in period t is in stage $s \in \{2, \ldots, S^{**}\}$, then in period $t + 1$ play is in stage $s - 1$. In period t is in stage 1, then in the next period play moves to stage $S^{**} + N$. In the other states (if $N > 0$), $s \in \{S^{**} + 1, \ldots, S^{**} + N\}$, the transition depends on the competitor's action. If he stays out, then the next period is in stage $s - 1$. If he enters, play jumps in the next period to stage $S^{**} - 1$.

Referring back to the finite-horizon equilibrium of the previous section, define S^{**} as $S^{**} = \min\{t : d_t < r\}$, S^* as $S^* = \min\{t : d_t > p^0 \text{ and } d_{t+1} < p^0\}$, and N as $N = \min\{n : M^{n+1}(0) > d_{S^*+1} \text{ and } M^{n+(S^{**}-S^*)+1}(1) < d_{S^*}\}$, where the value d_t is the cutoff belief in the t-th-to-last period of the finite-horizon equilibrium. That is, in the equilibrium of the finite horizon game, market S^{**} is the closest period to the end of the game in which the cutoff belief is below r, and market $S^* + 1$ is the closest in which the cutoff belief is below p^0. The value of N is large enough that both i) starting from a belief of 0, after the Markov process operates $N + 1$ times the updated belief exceeds $d_{S^*+1} < p^0$, and ii) starting from a belief of 1, $N + S^{**} - S^* + 1$ iterations of the Markov process drive beliefs below $d_{S^*} > p^0$. Strategies in each stage are described below. Beliefs on the equilibrium path come from Bayes' rule. If a competitor who is supposed to stay out enters, then beliefs are updated according to the incumbent's continuation strategy and the Markov parameters, just as in equilibrium. If an incumbent who is supposed to fight with probability one in period t accommodates, she is believed to be weak for sure, and so $p_{t+1} = M(0) = r$.

In stage $s \in \{1, \dots, S^{**}\}$, the incumbent fights with probability $\beta_s(p_t)$, where p_t is the current competitor's belief, and the function β_s gives the mixing probability in the s-th-to-last market of the equilibrium with finite T (given that $T \geq s$). The competitor enters if $p_t < d_s$ and stays out if $p_t > d_s$, where d_s is the cutoff belief in period s of the finite-horizon equilibrium. The competitor's probability of entry μ_s when he is indifferent (at $p_t = d_s$) must be modified slightly relative to the finite-horizon case to adjust for discounting by the incumbent: let $\mu_s = (\delta a - 1)/\delta a$. In state $s \in \{S^{**} + 1, \dots,$

$S^{**} + N\}$, the incumbent behaves after entry just as though the state were S^{**}: she fights with probability $\beta_{S^{**}}(p_t)$. The entrant stays out regardless of his belief.

In stage 1, the weak incumbent accommodates. That action is optimal because, by the definition of N, any posterior belief will deter entry until stage S^*, and no belief will deter entry longer. The competitor's strategy of entering in stage 1 when his belief is below d is thus a best response. The strategies in stages 2 through S^{**} are mutual best responses, just as they are in the finite-horizon case. (Note that $-1 + \delta[1 - (\delta a - 1)/\delta a]a = 0 + \delta \cdot 0$, so the incumbent is indifferent between fighting and accommodating today if tomorrow's competitor i) enters with probability $(\delta a - 1)/\delta a$ after observing fight today, and ii) enters for certain after observing accommodation.) In stages $S^{**} + 1$ through N, if the competitor enters, then the stage in the next period will be $S^{**} - 1$, and so the strategy for the incumbent of behaving just as in stage S^{**} is a best response. Since (by definition), $d_{S^{**}} < r$, then, the competitor, again as in stage S^{**}, prefers to stay out.

All that remains is to demonstrate that the incumbent's expected payoff in equilibrium is bounded below a for any discount factor. On the equilibrium path, competitors stay out in stages $S^* + 1$ through $S^* + N$, yielding the incumbent a payoff of a per period. The competitor enters in stage S^*, and by construction the expected payoff from then through stage 1 is 0. The incumbent's total discounted expected payoff, then, is

$$\frac{1-\delta}{1-\delta^{S^{**}+N}}\left(\frac{1-\delta^{S^{**}+N-S^*}}{1-\delta}a + \delta^{S^{**}+N-S^*}\frac{1-\delta^{S^*}}{1-\delta}0\right) = \frac{1-\delta^{S^{**}+N-S^*}}{1-\delta^{S^{**}+N}}a.$$

The coefficient on a is continuous in δ and is strictly less than 1 for all $\delta \in [1/a, 1]$. (The limit as δ approaches 1 is $(S^{**} + N - S^*) / (S^{**} + N)$.) Thus, the expected payoff to the incumbent with any discount factor is no greater than $a - \varepsilon$ for some $\varepsilon > 0$. *Q.E.D.*

If the incumbent is patient enough, then there is also an equilibrium in which she always fights, and thus entry is deterred in every period – the Stackelberg outcome. The threat of reversion to the equilibrium of Theorem 4 gives the weak incumbent incentive to fight. Corollary 5 formalizes that result.

Corollary 5: For generic values of the parameters *a*, *b*, *q*, and *r*, there exists a $\underline{\delta} < 1$ such that for any discount factor $\delta \in (\underline{\delta}, 1)$, there is a sequential equilibrium in which entry is deterred in every period and the expected payoff to the incumbent is *a*.

Proof: The strategies are for the competitor in every period to stay out, and for the incumbent to fight entry, as long as entry has never been accommodated. Beliefs are constant at p^0. After the first time the incumbent accommodates, play switches to the equilibrium described in Theorem 4, in stage $S^{**} + N$ with a belief equal to *r*. As long as $(1-\delta)(-1) + \delta a \geq (1-\delta)0 + \delta(a-\varepsilon)$, those strategies are an equilibrium. *Q.E.D.*

Thus, if the incumbent firm is patient, there are equilibria where she obtains Stackelberg payoffs, but it is never the case that those are the only equilibria.

6. Summary and Discussion

In the finite-horizon chain store game, the introduction of even very slight impermanence of the incumbent firm's type qualitatively alters the nature of the sequential equilibrium and the dynamics of reputation. Instead of maintaining a nearly-permanent reputation and discouraging potential entrants until the last few markets, the incumbent finds herself repeatedly tested by entrants, even when the last market is arbitrarily far in the future. A reputation for toughness deters competitors only temporarily; it must be continually refreshed. An effective reputation is one that deteriorates over time. If competitors believe with high probability that the incumbent is tough, then they will not enter, and thus will get no new information about the incumbent's type. The Markov process then drives beliefs down toward the stationary probability p^0.

On the other hand, in some periods (when the cutoff probability for deterrence is below r), entry is deterred even if the incumbent has just revealed herself as weak by accommodating in the previous market. That observation suggests that what drives deterrence is not so much a reputation for toughness – immediately after an accommodation the probability that the incumbent has become tough is very low – but rather a reputation for wanting to build a reputation for toughness, by fighting. That interpretation helps to explain the discontinuity between Selten's (1978) results and the results of Kreps and Wilson (1982) and Milgrom and Roberts (1982). Under Bayesian updating, it is impossible to revise a belief upward from zero. When the probability of the tough type is zero, the incumbent has no hope of establishing a reputation for toughness, and thus cannot have a credible reputation for wanting a reputation for toughness. In the model in this paper, in any period the probability that the incumbent is tough is bounded away from zero, and thus the incumbent can rebuild an effective reputation even after revealing herself as weak.

The qualitative difference in equilibrium between perfect and imperfect permanence results when the parameters of the Markov process (as well as the payoffs) are fixed, and the number of markets T is allowed to increase. Alternatively, T could be fixed and the Markov parameters allowed to approach perfect persistence. In particular, if the transition probabilities between types q and r shrink to 0 while their ratio (which determines the stationary distribution) and T remain constant, then it is straightforward to see from the derivation of equilibrium in Section 4 that equilibrium payoffs for all $T + 1$ players (and in fact the expected path of play) approach those from the permanent-type equilibrium: with arbitrarily high probability the incumbent's type does not change during the game, and since the function $W(\cdot)$ approaches the identity function, the cutoff beliefs and mixing probabilities converge to their permanent-type analogues.

When the number of markets T is limitless, again the set of equilibrium outcomes changes qualitatively when the incumbent's type becomes changeable. As with permanent types, there exists an equilibrium in which the incumbent always fights entry, and thus no competitor enters. Now, however, that equilibrium cannot be supported by the threat that all future competitors will enter if the incumbent ever reveals herself as weak by accommodating. Since a weak type may switch to being tough, that threat is no longer credible. Instead the Stackelberg outcome is supported by another equilibrium, of a sort that does not exist when types do not change. In it, play cycles between entry and deterrence, and the incumbent's expected payoff is strictly below her Stackelberg payoff a. That cycling equilibrium exists even when the incumbent is very patient. Thus, the reputation effects in Fudenberg and Levine (1989) that guarantee commitment payoffs to the long-run player when types are permanent become weaker when persistence is less than perfect. Fudenberg and Levine's (1989) proof relies on the insight that there is a bound on the number of times that entry can be fought before competitors become convinced that the incumbent will always fight. With changing types, though, beliefs do not stay above the necessary threshold. Again, an effective reputation deteriorates.

Finally, although for the sake of clarity and concreteness I have analyzed only the particular case of the chain store game, there is no obvious reason why similar results would not hold for other Stackelberg-type stage games. That observation suggests that existing results on reputation may not be robust to a relaxation of the assumption of perfect persistence. An extension of the model presented here to the infinite-horizon case is another interesting avenue for future research.

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Appendix: Strategic Tough Type

In this appendix, I return to the case of the strategic type of tough incumbent. In response to entry in the stage game, the strategic tough type can either fight or accommodate, but strictly prefers fighting to accommodating, and prefers no entry to fighting by the same margin a that the weak incumbent prefers no entry to accommodating. With a finite horizon, if the competitors are assumed not to revise their beliefs downward after seeing the incumbent fight, even off the equilibrium path, then the unique sequential equilibrium with a strategic tough type is the same as with a commitment type. That is, the strategic tough type always fights entry, and so the weak incumbent and the competitors behave just as they do with the commitment type.

Assumption A1, which gives the restriction on beliefs, is very similar in concept to the D1 refinement in the literature on signaling games. (See, for example, Section 11.2 of Fudenberg and Tirole (1991).) Roughly, the set of situations in which a weak incumbent finds it optimal to fight is a strict subset of those in which a tough incumbent wants to fight, and so competitors should not believe that a weak incumbent is more likely to fight than a tough one.

Assumption A1: For any period $t > 1$ and belief p_t that the incumbent is tough, if the period-t competitor enters and is fought, then the next period's belief $p_{t-1} \geq M(p_t)$.

Theorem A1: Let parameters a, b, q, r, and T be given, and suppose that Assumption A1 holds. Then in the generically unique sequential equilibrium of the game with a strategic tough type, the tough incumbent always fights entry, and the strategies of the weak incumbent and the competitors are the same as in the equilibrium with a committed tough type.

Proof: The proof is inductive. Let $\hat{\beta}_t(p)$, $\hat{d}_t$, and $\hat{\mu}_t$ denote respectively the values in equilibrium of the weak incumbent's probability of fighting entry in period t as a function of the current belief, competitor t's cutoff belief for entry, and competitor t's probability of entry when indifferent. Recall that $\beta_t(p)$, d_t, and μ_t are the equilibrium values derived in Section 4 for the game with a committed tough type.

Step 1: In the last market, a weak incumbent will accommodate entry, and a tough one will fight, so $\hat{\beta}_t(p) = 0 = \beta_t(p)$, and $\hat{d}_t = d = d_t$.

Step 2: Since fighting gives the tough incumbent a higher instantaneous payoff in period 2, and her payoff in market 1 is weakly increasing in p_1 regardless of her type in period 1, Assumption A1 ensures that the tough incumbent will fight entry.

The weak incumbent's incentives, then, are the same as in Section 4. Deterring entry creates a benefit equal to *a* in period 1 (whatever her type in that period) at an immediate cost of 1. Thus, she again will fight entry with probability $\hat{\beta}_2(p) = \beta_2(p)$, and competitor 1's probability of entering when indifferent $\hat{\mu}_1$ must equal μ_1. Note that the incumbent's expected payoff from period 2 on is weakly increasing in p_2, for both types.

Step t: Since her continuation payoff from market $t - 1$ on is weakly increasing in p_{t-1}, Assumption A1 again implies that the tough incumbent will fight entry. The weak incumbent and competitor *t* face the same situation as with a committed tough type (since again the gain to the incumbent of deterring entry is independent of her type), and so $\hat{\beta}_t(p) = \beta_t(p)$, $\hat{d}_t = d_t$, and if they exist $\hat{\mu}_{t-1} = \mu_{t-1}$.

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Thus, the tough incumbent fights entry in every market, and the strategies of the weak incumbent and the competitors are the same as those derived in Section 4. *Q.E.D.*

Note that the result that a strategic tough type always fights entry does not depend on the restriction that her gain in the stage game from deterring entry is exactly a. As long as that gain is strictly positive, Assumption A1 still implies that in equilibrium the incumbent always fights entry when she is tough. The equilibrium in that case is qualitatively similar to the commitment-type equilibrium, but the mixing probabilities of the competitors differ, because the gain to an incumbent from deterring entry in a future period now depends on her expected type in that period, which in turn depends on her current type and the number of periods before the deterrence.

In the infinite horizon case, it is straightforward to demonstrate that the equilibrium of Theorem 4 remains an equilibrium when the tough type is strategic – the tough type's strategy is to always fight entry. If, in addition, the tough type's payoffs are such that she gets a if there is no entry, 0 if entry is fought, and $c < 0$ if entry is accommodated, then it is not necessary to rescale the payoff bound in Theorem 4, and Corollary 5 goes through as before.