The Empirical Content of Models with Multiple Equilibria

Alberto Bisin Andrea Moro Giorgio Topa*

July 5, 2002 (VERY PRELIMINARY AND INCOMPLETE)

Abstract

We consider a generic environment with (potentially) multiple equilibria and analyze conditions that allow for the estimation of both the structural parameters and the "selected equilibrium". We focus on a "easy to compute" consistent 2-step estimator and use Monte Carlo methods to describe its finite sample properties.

1 Introduction

A widespread but undocumented belief among economists is that models with multiple equilibria do not have empirical content. It is sometimes claimed that using multiple equilibria anything can be explained, therefore such models cannot have empirical validation. The goal of this paper is to challenge this belief by suggesting an estimation procedure that delivers a consistent estimator of the fundamental parameters *and* of the "selected" equilibrium of a generic model with multiple equilibria.

There is a growing literature considering specific circumstances in which the parameters of a model with multiple equilibria can be estimated. Our focus, relative to such literature, is not only on the genericity of the environment we study, but also on the computability of the estimation procedure we propose.

Many economic models display multiple equilibria. Simple Arrow-Debreu endowment economies even with homothetic preferences can be constructed with any finite number of

*Bisin and Topa: New York University; Moro: University of Minnesota

equilibria (Debreu-Sonnenschein-Mantel theorem). With incomplete markets equilibria are in some instances indeterminate (that is, a continuum of equilibria exist) see Cass (contribution for the Econometric Society World Congress). In strategic environments multiple equilibria are the norm. Sufficient conditions for uniqueness in economies with externalities are very demanding (Glaeser-Scheinkman, 2001). Other applications arise in Macroeconomics (Cooper, 1999), Industrial Organization and in search models. We propose a framework that is generic enough to encompass most of these examples.

In principle, models with multiple equilibria can always be reduced to models with a unique equilibrium by appropriately expanding the set of parameters. This is essentially a tautological statement, to point out that the issue of empirical implications of such models is reduced to the more familiar question of identification of the extended set of parameters. Identification is in fact possible if one is willing to postulate some restriction on the selection mechanism operating over different realizations of the data generating process (e.g., if the parametrization of the equilibrium selection mechanism is independent of the number of observations, the realizations of the data generating process). Often, natural restrictions guarantee identification of the model jointly with the parametrization of the selection.

In this paper we consider a general set-up which allows data to be realized from one or more of the feasible equilibria. We show that even if the parameters (including the selection parameters) are identified, when there are multiple equilibria the estimation of the structural parameters is in general a daunting computational task. The likelihood of the data can only in fact be defined conditionally on the equilibrium selection. To correctly compute the likelihood, one has to be able to compute all of the equilibria that are consistent with a given set of parameters. If this is possible, a "direct estimator" of the structural parameters can be computed by maximizing the likelihood over both the set of equilibria and the set of the structural parameters. However, except in the simplest models, the dimensionality of the equilibrium set is not known ex-ante nor is it constant across the parameter space, which makes this approach computationally very difficult.

Therefore, we suggest an estimator that considers the "equilibrium" (more generally, a vector of variables that are sufficient statistics for the description of the equilibrium), as an additional estimable parameter. We propose a two-step method that, in the first step, estimates the equilibrium together with the other structural parameters without using to the equilibrium restrictions imposed by the model. Such equilibrium restrictions are imposed in the second step, to recover the structural parameters that are consistent with the equilibrium estimated in the first step, which is now taken as given. This method is computationally easier, given that it does not require the computation of all of the feasible equilibria.

We show that the two-step estimator is consistent, but in finite sample is equivalent to the direct estimator only under a set of sufficient conditions with are very restrictive. Finally, we use Monte Carlo simulations to analyze small sample properties of the estimators we propose.

2 Related Literature

In an early contribution Jovanovic (1989) focuses on general identification conditions for the estimation of models with multiple equilibria but does not discuss the technical aspects of the estimation. Dagsvik-Jovanovic (1994) study economic fluctuations in a model with two equilibria (high and low economic activity); they postulate a stochastic (Markovian) equilibrium selection process over time and estimate the parameters of such process with time series data on economic activity. The adopted functional form specification allows the investigator to derive closed form solutions of the “inverse equilibrium map” (the mapping from the set of equilibria to the set of parameters), which helps contructing the sample likelihood. Such map is assumed to be a function.

Others (for example: Bresnahan and Reiss 1991, Tamer 2001) consider games of complete information where the investigator only observe the action played by the agents, whereas the parameters to be estimated affect also the payoffs. Clearly, in this case the inverse equilibrium map cannot be a function (a continuum of parameter values is consistent with the same equilibrium realization of the strategy profile). However, it is possible to find identification conditions whenever multiplicity is associated only with a subset of the feasible parameter values, but not with the entire set. In this case one can in some sense exploit the multiplicity to recover identification.

Brock and Durlauf (2001) consider models of social interaction and consider situations where the data realization depends not only on the "selected equilibrium" but also on the fundamental parameters of the model. They derive conditions for identification which do not rely on assuming a specific equilibrium selection device.

In other cases it is natural to assume that all the observations are generated at the same equilibrium. Moro, 2001, studies a model of statistical discrimination across racial characteristics in which multiple equilibria exists (high and low human capital investments by racial group). The equilibrium map linking wages to individual characteristics is different across different equilibria and hence the equilibrium selection can be identified off cross-sectional data. Notice that in this case, only one realization of the equilibrium is observed and identification is obtained by exploiting the properties of the equilibrium map. The ability to identify and estimate models with multiple equilibrium in which only one realization of the equilibrium is observed is an important innovation contributed by this paper: Dagsvik-Jovanovic, for instance, can observe several equilibrium realization only because they study a static model (repeated over time); had they modelled a dynamic economy (e.g., had they modelled capital accumulation) they would have found themselves with a single equilibrium realization.

[...to be continued...]

3 The Setup

Consider an economy populated by a finite set of agents indexed by $i \in I$. Each agent i is endowed with preferences represented by the utility function

$$U^i\left(x^i, \mathbf{x}^{-i}; \theta', u'\right)$$

where x^i represents a vector of unspecified arguments of utility in a general compact set X, the index $-i$ denotes the set of all agents except i, and $\mathbf{x}^{-i}$ stacks all vectors x^j for $j \neq i$; also θ' denotes a vector of preference parameters and u' a random vector. Note that the specification of preferences allows for strategic interactions across agents.

Each agent $i \in I$ chooses x^i to maximize his utility given $\mathbf{x}^{-i}$ and a vector of endogenous variables p defined in a compact set P, which appears in the constraint set, $X^i(p, \mathbf{x}^{-i}; \theta'', u'')$; where θ'' denotes a vector of parameters, and u'' a random vector. We let $\theta \equiv [\theta', \theta'']$ defined in some compact Θ and $u \equiv [u', u'']$ have support in some compact U, with p.d.f. $f(\cdot)$. The agent's problem:

$$\max_{x^i \in X^i(p, \mathbf{x}^{-i}; \theta'', u'')} U^i \left(x^i, \mathbf{x}^{-i}; \theta', u'\right) \tag{1}$$

Assumption 1 *For any agent $i \in I$:*
$U^i \left(x^i, \mathbf{x}^{-\mathbf{i}}; \theta\prime, u\prime\right)$ *is smooth in all the arguments, and strictly concave in x^i;*
$X^i(p, \mathbf{x}^{-\mathbf{i}}; \theta\prime\prime, u\prime\prime)$ *defines a convex valued continuous correspondence mapping $(p, \mathbf{x}^{-i}, u\prime\prime)$ into X.*

From Assumption 1 and Berge's maximum theorem, it follows that the solution of problem (1) is represented by a continuous function mapping $(p, \mathbf{x}^{-i})$ into x^i, which we write

$$x^i = x^i \left(p, \mathbf{x}^{-i}; \theta, u\right)$$

Let $\mathbf{x}(p, \mathbf{x}; \theta, u)$ denote the composition mapping, from $x^i = x^i \left(p, \mathbf{x}^{-i}; \theta, u\right)$, over i. Let F denote a vector valued mapping defined on $\pi \equiv (p, \mathbf{x})$.

Assumption 2 *The mapping F is smooth in all its arguments.*

Definition 1 *An equilibrium of the economy is a vector $\pi \equiv (p, \mathbf{x})$ such that*

$$F\left(p, \mathbf{x}(p, \mathbf{x}; \theta, u)\right) = 0 \tag{2}$$

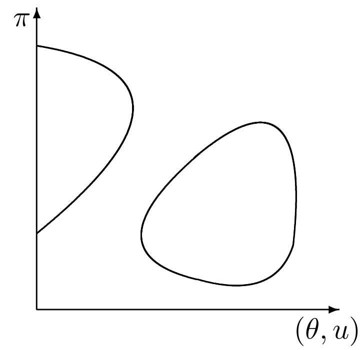

Figure 1: A model without global identification

Under Assumptions 1 and 2, with some extra regularity and dimensionality assumptions, the equilibrium can be represented in general as a map from (θ, u) into π which has the property of a smooth manifold.

Let $\pi(\theta, u)$ be such a map. Assumptions guaranteeing that $\pi(\theta, u)$ is one-to-one and defined for all $\theta \in \Theta$ are extremely restrictive, and frequently not satisfied in economic models (see Figure 1 for a manifold which does not satisfy either of these properties) .

Under Assumptions 1 and 2, and some extra regularity and dimensionality assumptions, the equilibrium can also be represented in general as a map from (π, u) into θ, which also has the property of a smooth manifold. Moreover, we assume the following.

Assumption 3 $\theta(\pi, u)$ *is single valued, for* π *in a subset (possibly strict) of its domain* $P \times X$*, and not defined everywhere else.*

In other words, for any vector π and any u, either one and only one parameter θ exists such that π is an equilibrium, or none. This assumption is essentially a (global) identification condition,[1] and is required even if the economy does not display multiple equilibria. In Figure 2 we show a manifold $\pi(\theta, u)$ which in not one-to-one (as an equilibrium manifold it displays multiple equilibria), but is such that the associated manifold $\theta(\pi, u)$ is single valued, and hence satisfies Assumption 3).

[1] Formally, $\theta(\pi, u)$ needs to be single valued only in a neighborhood of the global maxima of the likelihood.

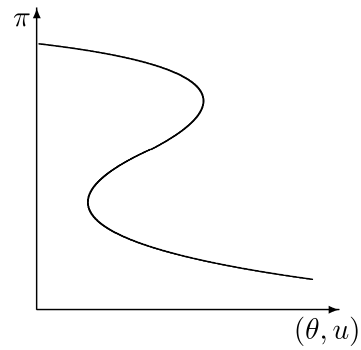

Figure 2: A model with multiple equilibria and global identification

4 Observations

Let $y \in Y$ denote a vector of observable variables. We assume there exists a map g from the equilibrium variables π, parameters θ and random vector v into y. The random vector v is a vector of disturbances drawn from distribution $h(v)$ affecting the observations but not the equilibrium (for example, observation errors).

Assumption 4 *Fix $\theta \in \Theta$ and a realization v; the map $g(\pi, \theta; v)$ is smooth, one-to-one, and onto in π.*

5 Maximum Likelihood Estimator

In general, the likelihood function of θ for the random variable y is defined as

$$L(y|\theta) \equiv p(y; \theta)$$

where $p(\cdot)$ is the p.d.f. of y. In our setup, given the possible presence of multiple equilibria, the likelihood $L(y|\theta)$ is in general a correspondence, and often quite complex to compute. Such likelihood is defined as follows:

$$L(y|\theta) = \int_{(u,v):g(\pi(\theta,u),\theta,v)=y} h(v)f(u)dvdu \tag{3}$$

where $\int$ denotes the Aumann integral[2] and $\pi(\theta, u)$ satisfies the equilibrium condition (2). Loosely speaking, the Riemann integral is not defined since $\pi(\theta, u)$ is in general a correspondence; the Aumann integral is defined for correspondences and is constructed by taking the union of the Riemann integrals of all measurable selections of the correspondence; it coincides with the Riemann integral when applied to a measurable function.

Let $L(\cdot, \theta)$ be the set of probability distributions over the realization of y given θ defined by (3). Then, in our setup, generalizing the standard definition of identification we have that the parameter vector θ_0 is identified if for all $\theta \in \Theta$, $\theta \neq \theta_0$, $\forall\, l \in L(\cdot\,|\,\theta_0)$, $l \notin L(\cdot\,|\,\theta_1)$.

Proposition 2 *Under Assumptions 3 and 4, θ is in identified*

5.1 A direct method

Suppose one observes a random sample $\mathbf{y} \equiv (y_1, ..., y_N)$. The sample likelihood function based on (3) is

$$L(\mathbf{y}|\theta) \equiv \frac{1}{N} \prod_{i=1}^{N} \log L(y_i|\theta)$$

The direct estimator of θ can then be defined as follows:

$$\widehat{\theta} = \arg\max_{\theta} L(\mathbf{y}|\theta). \tag{4}$$

Because of the possible multiplicity of equilibria, $L(\mathbf{y}|\theta)$ is very difficult to characterize, as the maximum must be taken over the parameter space jointly with all the admissible integrable selections of the correspondence $L(\mathbf{y}|\theta)$. However not all integrable selections need to be considered; in particular for each realization of the parameter vector, only the equilibrium that maximizes the likelihood (over the set of feasible equilibria) should be considered.

Define $L(\mathbf{y}|\pi, \theta, u)$ the likelihood conditional on equilibrium π being realized when the

[2] See Aliprantis for the formal definition and a discussion of the properties of such integral.

shock is u:

$$L(\mathbf{y}|\pi,\theta,u)=\begin{cases} h(v) & \begin{array}{l} if \quad F\left(p,\mathbf{x}(p,\mathbf{x};\theta,u)\right)=0 \\ \text{where } v \text{ satisfies } g(\pi,\theta;v)=\mathbf{y} \end{array} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Formally, (4) is equivalent to:

$$\widehat{\theta}=\arg\max_{\theta}\left(\max_{\pi}\int\limits_{u:F(\pi;\theta,u)=0}L(\mathbf{y}|\pi,\theta,u)f(u)du.\right) \tag{6}$$

where v satisfies $y=g(\pi,\theta,v)$.

Proposition 3 *The direct estimator $\widehat{\theta}$ is consistent and efficient.*

Proof. [3]Define, for any equilibirum, the following conditional likelihood:

$$L(\theta;\pi)=\int_{u:F(\pi;\theta,u)=0}L(y|\pi^{j},\theta,u)f(u)du$$

(If $F(\pi,\theta,u)=0$ is never satisfied then $L(y|\pi^{j},\theta,u)=0$). Notice that $L^{j}(\theta)$ is well defined likelihood since we are conditioning on π so we have a well defined probability distribution over y. If we knew the equilibrium, say π_0, we could maximize $L(\theta;\pi_0)$ to obtain an estimator $\hat{\theta}(\pi_0)$. With typical regularity conditions this estimator is consistent, asymptotically normal and efficient. Now define the following estimator:

$$\hat{\theta}_{MLE}=\hat{\theta}(\pi)\Leftrightarrow\max_{\pi}\max_{\theta}L(\theta;\pi)$$

Since asymptotically we choose $\hat{\theta}_{MLE}=\hat{\theta}(\pi_0)$ with probability one, then $\hat{\theta}_{MLE}$ inherits its properties. Finally, note that

$$\hat{\theta}_{MLE}=\max_{\pi}\max_{\theta}L^{j}(\theta;\pi)=\max_{\theta}\max_{\pi}\int_{u}L(y|\pi^{j},\theta,u)f(u)du=\hat{\theta}$$

■

[3]We thank Victor Aguirregabiria for suggesting this proof strategy.

The estimator in (6) requires, for each (θ, u), to compute all of the feasible equilibria, compute their likelihood, choose the maximum, integrate over u and maximize over θ. Such procedure is computationally difficult to implement especially when the number of equilibria is not known. because it may miss one or more equilibria. This is particularly relevant in the situations when the parametric form of $F(\pi; \theta, u)$ does not allow the investigator to know in advance how many solutions the equilibrium correspondence displays.

5.2 A Two-Step method

We now introduce a two-step estimation procedure. The first step consists in computing an estimator that ignores the equilibrium restriction from (2) in (5) and considers π as an additional parameter to estimate. Define

$$\begin{aligned} L(\mathbf{y}|\pi, \theta) \quad &\equiv \quad \int_u h(v) f(u) du \\ &\quad\;\; \text{where } v \text{ satisfies } g(\pi, \theta; v) = \mathbf{y} \end{aligned} \tag{7}$$

The first step solves:

$$(\widehat{\pi}_1, \widehat{\theta}_1) = \arg\max_{\pi,\theta} L(Y|\pi, \theta) \tag{8}$$

It is important to notice that the equilibrium restriction $\pi = \pi(\theta, u)$ is not imposed and both π and θ are treated as free parameters.

In the second step, one re-estimates θ taking $\pi = \widehat{\pi}_1$ as given, but imposing the equilibrium restriction, to take into account the equilibrium conditions:

$$\widehat{\theta}_2 = \arg\max_{\theta} \left(\int\limits_{u:\theta=\theta(\widehat{\pi}_1, u)} L(\mathbf{y}|\widehat{\pi}_1, \theta, u) f(u) du. \right). \tag{9}$$

Note that the right hand side in (9) is the same as in (6) once the first-step estimated equilibrium $\widehat{\pi}_1$ is substituted in. The estimated values $\left(\widehat{\pi}_1, \widehat{\theta}_2\right)$ satisfy the equilibrium restrictions by construction, but the estimation does not require the computation of all the equilibria, as $\pi = \widehat{\pi}$, is taken as given, and $\theta(\pi, u)$ is well-behaved by Assumption 3.

5.3 Equivalence Between Estimators

We can then prove the following proposition describing two alternative sets of sufficient conditions for equivalence to hold.

Proposition 4 *Sufficient conditions for equivalence of the two-step and the direct estimation of π procedures are either of the following:*

1. *for any (π, θ) there exists a unique u such that $\pi = \pi(\theta, u)$; moreover, $u \sim$ uniform*

2. *$\theta(\pi, u)$ is independent of u and takes a (unique by Assumption 3) value for any π*

Proof. Condition 1) states that there is always a u such that π is an equilibrium given parameters θ. But then if u is uniform $f(u)$ is a constant, therefore the second step is redundant and any maximizer of (8) is also a maximizer of (6).

Conditions 2) state that the realization of the data depend only on π, not on θ. Hence in the first step only π is identified. The second condition states that the model is non-stochastic, and that the mapping from π to θ is a function. Hence, after having estimated π in the first step, it is possible to uniquely recover an estimate of π in the second step ■

Moro (2001) is the first to employ the 2-step procedure to estimate a model with multiple equilibria. In his model condition 2 holds and therefore the equivalence of the two procedures follows readily.

It is interesting to consider a relaxation of condition 2). Suppose $\theta(\pi, u)$ is independent of u, but, in accordance with Assumption 3, it takes a unique value π for π in a subset (possibly strict) of its domain $P \times X$, and is not defined everywhere else. In this case equivalence does not hold, but the two step procedure can be easily modified to guarantee equivalence. The appropriate modified 2-step procedure requires jointly estimating (π, θ) by:

$$\begin{pmatrix} \widehat{\pi} \\ \widehat{\theta} \end{pmatrix} = \arg\max_{\pi,\theta} \int_u h(v) f(u) du \text{ such that } \theta = \theta(\pi)$$

5.4 Consistency of the Two-Step Estimator

We can now discuss the asymptotic properties of each of the two-step estimator outlined above.

Proposition 5 *The two-step estimator of* (π, θ) *is consistent if*
1) $L(Y|\pi, \theta)$ *has a unique maximum in* (π, θ);
the estimator of π *is consistent if*
2) $L(Y|\pi, \theta)$ *has a unique maximum in* π *which is independent of* θ.

Note that in the second case, consistency for θ is meaningless, as by construction we only observe one realization of π which we can use to estimate θ. In the next section we will study economies in which multiple realization of π are observed and hence the issue of consistency of our estimator for θ can and will be addressed.

[...to be continued...]

6 Multiple Realized Equilibria

In the analysis so far, we have implicitly assumed that, while there are multiple equilibria, a single *realized* equilibrium holds for the entire economy. In general, however, we may be interested in situations in which different units within the economy may find themselves in different equilibria. Think, for example, of a set of segmented markets within the economy. While the fundamental parameters are the same, in each market a different equilibrium is potentially selected. Suppose there are N markets in the economy, which can find themselves at distinct equilibria. We wish to discuss the direct estimator $\left(\widehat{\theta}, \widehat{\pi}\right)$ and the two-step estimator $\left(\widehat{\pi}_1, \widehat{\theta}_2\right)$ in this context.

One complication is that we do not know ex-ante how many distinct equilibria there are. Allowing markets to be in different equilibria, we can repeat the estimation procedures defined above (both the direct and the 2-step method) and use (5) or (7) to define each market's contribution to the joint likelihood. Notice that the existence of multiple markets at potentially different equilibria provides additional identification of the fundamental parameters vector θ.

More interesting is the case where the equilibria are correlated across markets. For example, one can think of a situation where neighboring cities are more likely to be at the same equilibrium than distant cities. In a dynamic environment modeled as a repeated static model, one can assume correlation between selected equilibria over time. In this case, the parameters defined by the correlation have to be jointly estimated with the other fundamentals.

[...to be continued...]

7 Examples

We now apply the estimation procedure to three example adapted from existing literature. The first is a version of a generic model of global interaction from Brock and Durlauf (2001).

7.1 A Global Interaction model

Consider a world in which there are N cities, I agents per city. Agents are characterized by a scalar characteristic X_i, observed by the econometrician. Cities are characterized by a shock $u_{\ n}$, unobserved to the econometrician. Agents choose an outcome $y_i \in \{-1, 1\}$. We assuming that the individual's payoff depends on her expectation about the choice of the other agents $E_i(y_{-i})$ within each city. The individual choice y_i is the solution to

$$\max_{y_i} V\left(y_i, X_i, u_n, E_i(y_{-i}), \varepsilon_i(y_i)\right),$$

where ε_i are individual errors shocks.

We specialize the model first by assuming that the ε_i are extreme value distributed which implies that the difference $\varepsilon_i(-1) - \varepsilon_i(1)$ is logistically distributed:

$$\Pr\left(\varepsilon_i(-1) - \varepsilon_i(1) \leq z\right) = \frac{1}{1 + \exp(-\beta z)};$$

Secondly, we assume that payoff can be additively decomposed into three terms:

$$V\left(y_i, X_i, u_n, \varepsilon_i(y_i)\right) = h(X_i, u_n) \cdot y_i + \sum_{j \neq i} J_{ij} y_i E(y_j) + \varepsilon_i(y_i).$$

with a linear specification for $h(X_i, u_{\ n})$:

$$h(X_i, u_n) = k + cX_i + u_n.$$

Finally, we assume that the agents interaction depends only on average behavior within each city i.e. $J_{ij} = \frac{J}{I}$, Then, (**??**) becomes

$$V\left(y_i, X_i, u_n, \varepsilon_i(y_i)\right) = (k + cX_i + u_n) \cdot y_i + J y_i \pi_n + \varepsilon_i(y_i), \tag{10}$$

where $\pi_{\ n} = E(\overline{y}_n)$, the expectation of the average action in city n. Assuming rational expectations, the expected value of each individual choice is constrained by self-consistency conditions which imply that the equilibrium average choice $\pi_{\ n}$ in city n is determined by

$$\pi_n = \int \tanh\left(k + cX + u_n + J\pi_n\right) dF(X). \tag{11}$$

Solutions to (11) correspond to city-wide equilibria. It can be shown that for the equilibrium is unique if $J < 1$, otherwise if $J > 1$ there are three distinct equilibria.

7.1.1 Estimation

The probability that agent i makes choice y_i is equal to the probability that the utility of y_i is greater than the utility of $-y_i$:

$$\begin{aligned}\Pr\left(y_i | X_i, u_n, \pi_n\right) &= \Pr\left(V\left(y_i, X_i, u_n, \varepsilon_i(y_i)\right) > V\left(-y_i, X_i, u_n, \varepsilon_i(-y_i)\right)\right) = \\ & \Pr\begin{pmatrix} (k + cX_i + u_n) \cdot y_i + J y_i \pi_n + \varepsilon_i(y_i) > \\ -(k + cX_i + u_n) \cdot y_i - J y_i \pi_n + \varepsilon_i(-y_i) \end{pmatrix}.\end{aligned}$$

One can show that the logistic specification of the errors implies that

$$\Pr\left(y_i = 1 | X_i, u_n, \pi_n\right) \sim \exp\left((k + cX_i + u_n) \cdot y_i + J y_i \pi_n\right).$$

(obviously $\Pr(y_i = -1|X_i, u_n, \pi_n) = 1 - \Pr(y_i = 1|X_i, u_n, \pi_n)$. Since the random utility terms are independent across individuals, one obtains, for the vector of choices y_n within city n:

$$\begin{aligned} \Pr(\mathbf{y}_n|\mathbf{X}_n, u_n, \pi_n) &= \prod_i \Pr(y_i|X_i, u_n, \pi_n) \sim \\ & \prod_i \exp\left((k + cX_i + u_n) \cdot y_i + Jy_i\pi_n\right). \end{aligned} \tag{12}$$

Equation (12) suggests the following formulation of the likelihood function as a function of the parameter vector $\theta = \{k, c, J\}$ $L(y_n|\mathbf{X}_n, u_n, \pi_n; \theta)$:

$$\begin{aligned} L(\mathbf{y}_n|\mathbf{X}_n, u_n, \pi_n; \theta) &= \prod_i \left[\Pr(y_i = 1|X_i, u_n, \pi_n)\right]^{\frac{1+y_i}{2}} \cdot \left[\Pr(y_i = -1|X_i, u_n, \pi_n)\right]^{\frac{1-y_i}{2}} \sim \\ & \prod_i \left[\exp(k + cX_i + u_n + J\pi_n)\right]^{\frac{1+y_i}{2}} \cdot \left[\exp(-k - cX_i - u_n - J\pi_n)\right]^{\frac{1-y_i}{2}} \end{aligned} \tag{13}$$

The Direct Method

If an estimator satisfying the equilibrium condition is important for the analyst, a brute force approach consists in using (5) and estimate

$$\widehat{\theta} = \arg\max_{\theta} \prod_n \left(\max_{\pi_n} \int_{u_n : \int \tanh(k + cX + u_n + J\pi_n) dF(X) - \pi_n = 0} L(\mathbf{y}_n|\mathbf{X}_n, u_n, \pi_n; \theta) f(u_n) du_n. \right)$$

Such an estimator is computationally very expensive: for each θ, u , all equilibria π_n consistent with θ and u have to be computed.

A Naive Estimator

Brock and Durlauf (2001) consider π_n as known and suggest a "naive" estimator of the fundamental parameters vector $\{k, c, J\}$ based on the maximization of the likelihood defined in (13):

$$\widehat{\theta}_n = \arg\max_{\theta} \prod_n \int_{u_n} L(\mathbf{y}_n|\mathbf{X}_n, u_n, \pi_n; \theta) f(u_n) du_n. \tag{14}$$

. While the estimator is consistent, nothing guarantees that with a finite sample the solution satisfies the equilibrium restriction (11) for a particular value of $\pi_{\ n}$. This correspond to the first step of our two-step method

The 2-step Method

To guarantee that the equilibrium restrictions are satisfied, we consider, in the first step, the estimation of the equilibrium $\{\widehat{\pi_{n1}}\}_{n=1}^{k}$ in each city as a by-product of the naive estimator:

$$\{\widehat{\pi_{n1}}\}_{n=1}^{k} = \arg \max_{\theta, \{\pi_n\}_{n=1}^{k}} \prod_n \int_{u_n} L(\mathbf{y}_{\ n}|\mathbf{X}_n, u_n, \pi_n; \theta) f(u_n) du_n$$

and then, in the second step, impose the equilibrium restriction:

$$\widehat{\theta}_2 = \arg \max_{\theta} \prod_n \int_{u_n : \int \tanh(k + cX + u_n + J\widehat{\pi_{n1}}) dF(X) - \widehat{\pi_{n1}} = 0} L(\mathbf{y}_n|\mathbf{X}_n, u_n, \widehat{\pi_{n1}}; \theta) f(u_n) du_n.$$

Notice that the restriction under the integrand is "easy" to compute since for each parameter vector θ and equilibrium $\pi_{\ n}$ there is a unique u_n satisfying the restriction.

8 Simulations

[...to be continued..]

References

Aguirregabiria, Victor, "A Two-Stage Estimator for Discrete Games of Incomplete Information", mimeo, Boston University, 2001.

Bisin, Alberto, Giorgio Topa, and Thierry Verdier, "An Empirical Analysis of Religious Homogamy and Socialization in the U.S.," mimeo, NYU, 2000.

Breshnahan, T. F. and P. C. Reiss, "Empirical Models of Discrete Games", *J. of Econometrics* **64,** 57-81, 1991

Brock, W. A. and S. N. Durlauf, "Interaction-Based Models," in J. J. Heckman and E. E. Leamer, eds., *Handbook of Econometrics* (North-Holland, 2001), Chapter V.

Cooper, R., *Coordination Games: Complementarities and Macroeconomics.* (Cambridge University Press, 1999).

Dagsvik, J. and B. Jovanovic, "Was the Great Depression a Low-Level Equilibrium?" *European Economic Review* 38 (December 1994), 1711–29.

Gallant, *Introduction to Econometric Theory.*

Glaeser, Edward, and Jose' A. Scheinkman, "Measuring Social Interaction," mimeo, Harvard, 2000.

Heckman, James, "Sample Selection Bias as a Specification Error with an Application to the Estimation of Labor Supply Functions," in James P. Smith (Ed.), *Female Labor Supply: Theory and Estimation*, Princeton University Press, 1980.

Jovanovic, Boyan, "Observable Implications of Models with Multiple Equilibria", *Econometrica*, 57 (1989), 1431-1437.

Moro, Andrea, "The Effect of Statistical Discrimination on Black-White Wage Inequality: Estimating a Model with Multiple Equilibria," mimeo, University of Minnesota, 2001.

Pakes, Ariel, and David Pollard, “Simulation and the Asymptotics of Optimization Estimators”, *Econometrica*, 57 (1989), 1027-1057.

Tamer, Elie, “Incomplete Simultaneous Discrete Response Model with Multiple Equilibria,” mimeo, Princeton, 1999.