Pricing Rare Event Risk in Emerging Markets

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Abstract

This paper solves the pricing problem of an emerging market debt contract in which the borrower's economy is subject to rare event risk. Our model combines elements of a reduced-form and a structural model of debt pricing. Rare event risk is modeled as a sudden event in fundamentals, and we study the role of the debt contract in providing risk sharing between the borrower and the lender. The two main frictions under consideration in our equilibrium model are limited participation of the lender through the debt contract, and heterogeneous beliefs between the borrower and the lender about the likelihood of a rare event. We solve for the rate of interest, the credit spread, the risk premium, the write-off (recovery rate) in case of default, and the dynamics of the debt contract in non-default times. We find that limited participation combined with heterogeneous beliefs has strong effects on the level and variability of the debt contract properties.

Keywords: Rare Event Risk, Emerging Markets, Exchange Economy, Jump-Diffusion Model, Heterogeneous Beliefs, Incomplete Market.

JEL Classification: D51, D52, E43, F34, G12.

1 Introduction

Emerging market economies are often subject to rare event risk which can severely affect their aggregate productivity. We think of rare event risk as risks originating from natural or man-made catastrophes, terrorism, or risks due to political instability.[1] In this paper we provide an equilibrium solution to the pricing problem faced by a lender, for example an international financial institution, the IMF, or the Worldbank, on an investment decision in an emerging market country subject to rare event risk through a debt contract.

Our pricing exercise is based in the following primitives. We model a borrower's economy that is subject to rare event risk in addition to regular economic risk. Both types of risk are systematic. A risk averse lender can only invest in the borrower's economy through a debt contract, which is consistent with the very low levels of foreign direct investment observed in emerging markets. Furthermore, the borrower and the lender might not agree on the likelihood of a rare event which is a natural assumption since the true frequency of rare events is unknown. Hence, the two main frictions of the model are limited participation and heterogenous beliefs. The purpose of our work is to study how these frictions affect the pricing of a risky debt contract in an equilibrium setting.[2] We fully specify the dynamics of the debt contract in terms of economic fundamentals, in particular the rate of interest, the credit spread, the risk premium, the write-off (recovery rate) in case of default, and the dynamics of the debt contract in non-default times. The dynamic nature of our model also allows us to describe how the borrower and the lender are affected in terms of their financial wealth, and how the interest rate and the write-off adjust at the occurrence of default.

[1] A recent example of a natural catastrophe with severe a impact was triggered on December 26, 2004, when a magnitude 9.0 earthquake occurred off the west coast of Sumatra, Indonesia. This was the fourth largest earthquake in the world since 1900. The earthquake generated tsunamis which swept across the Indian Ocean. The worst affected country was Indonesia - Aceh province; over 120,000 people lost their lives in this disaster. The World Bank has estimated total economic damages and losses caused by the earthquake and tsunami at approximately US$ 4.45 billion, or almost 100 percent of Aceh's GDP in 2003. The tsunamis also affected Phuket and surrounding areas in Thailand, Malaysia, Sri Lanka, India, and places in Africa.

[2] In the Appendix, we show how to relax the assumption of zero foreign direct investment (FDI). In particular, we discuss how the interest rate, write-off, and the variability of the credit spread is affected once the lender diverts a small amount of her investment directly into the productive assets of the borrower.

To our knowledge, we are the first to study the effect of heterogeneous beliefs about rare events on the pricing of risky debt. We find that the debt's interest rate is decreasing in the belief of the lender, but increasing in the belief of the borrower. The impact of heterogeneity can be large enough such that the borrower is willing to pay an interest rate to the lender which is significantly higher than the fundamental's growth rate, as often observed for emerging market economies.[3] Both directions are linked to the endogeneity of the optimal write-off on the debt contract in case of default.[4] The debt write-off is decreasing in the belief of the borrowing agent, but increasing in the belief of the lending agent.

Our model delivers strong effects on the variability in non-default times. It generates stochastic interest rates, credit spreads, and risk premiums. We show how a levered economy combined with heterogeneous beliefs leads to equilibrium volatilities possibly several times higher than under homogeneity. Under belief homogeneity, we would not observe any variation in credit spreads and the risk premium, and the (small) variation in the interest rate is entirely due to variation in the "shadow" riskless interest rate.

We calibrate the model to economic fundamentals and historical credit spreads of Ecuador. The country defaulted in August 1999 due to rare event risk and completed an exchange offer for their external debt in July 2000. Our results suggest that recovery rates (write-off values) on external debt, interpreted solely from a risk sharing perspective, should have been higher (lower) than ex-post observed values. While we are not claiming that heterogeneity in beliefs and leverage should explain all observed volatility in credit spreads, our results do, however, suggest that a moderate degree of dispersion can explain a significant fraction of Ecuador's observed volatility. Since debt pricing models generally neglect heterogeneous beliefs, it is unknown to what extent this friction can contribute to the stochastic nature of observed prices.

[3] In Table 1, we show estimates of the growth rate of aggregate output for the emerging market economies Ecuador, Indonesia, Panama, and the Philippines to be 5.00%, 5.03%, 4.47%, and 5.13%, respectively. The mean real interest rates for these economies for the same time period are 11.11%, 8.70%, 9.77%, and 5.13%, respectively, based on annual observations according to the Global Development Finance and World Development Indicators.

[4] Most sovereign debt defaults lead to ex-post negotiations about debt recovery in the absence of a sovereign bankruptcy code. Our model suggests debt-write off values solely from the perspective of risk sharing. It does therefore carry potential implications important for the ongoing discussion, how a sovereign bankruptcy framework should be structured, see Bolton and Skeel (2004).

Our pricing approach builds on the limited market participation model of Basak and Cuoco (1998), since a lender participates in the borrower's economy only through a debt contract.[5] Once the borrower's economy faces rare event risk in addition to regular economic risk, riskless borrowing becomes infeasible. The intuition for this result can be derived even from a partial equilibrium perspective. Suppose a price system faces rare events, and a levered portfolio optimizer is one jump away from ruin, then he will not be able to quickly reverse his portfolio in order to avoid negative wealth. Hence, as shown for example by Liu, Longstaff and Pan (2003), the optimal portfolio is one where the investor does not engage in a levered position. Taking this intuition to our equilibrium pricing exercise in a levered economy that does face rare event risk, this in turn implies that the debt contract can not be riskless, but should allow for a risk sharing opportunity when a rare event occurs.

In light of the previous result, it should not seem surprising that many countries facing rare events default on outstanding debt contracts. In fact, it is optimal for the borrower to do so, and to shift some exposure of the impact of a rare event to the lender. Four recent examples from emerging market countries, in which a dramatic exogenous shock corresponded to default on debt, are Ecuador, Indonesia, Panama, and the Philippines.[6] To motivate the dynamics of our exogenous state variable, we carry out the following estimation. In a continuous-time formulation, real GDP is assumed to follow a jump diffusive process given by

$$\frac{de(t)}{e(t-)} = \mu_e dt + \sigma_e dB(t) + \kappa_e dN(t, \lambda),$$

where $\mu_e, \sigma_e, \kappa_e$, and λ serve as the deterministic growth rate, the volatility parameter, the jump size, and the jump intensity, respectively. Estimation is carried out via maximum likelihood. The data set consists of yearly observations between 1974 and 2003 for the countries mentioned. The results are displayed in Table 1, and the estimation procedure clearly identifies the extreme shock, with an insignificant low frequency. For example, according to this

[5]Basak and Cuoco (1998) analyze properties of risk premia in the stock market given a levered economy, while our extension with rare event risk analyses the properties of the debt contract given a levered economy.

[6]More detailed country-specific information about defaults and recovery rates can be found in Moody's (2003) and Standard and Poor's (2002). Details on the case of Ecuador can be found in Section 4. For the case of Indonesia, there was a sovereign debt default, a banking collapse, and mass corporate bankruptcy in 1998.

estimator, Ecuador's GDP exhibited a negative jump of -16.77%. For all of these countries, the identified negative jump led to default on outstanding debt.

	μ_e	σ_e	κ_e	λ
Ecuador	.0500 (7.7)	.0437 (8.8)	$-.1677$ (-3.3)	.0302 (1.0)
Indonesia	.0503 (9.5)	.0380 (10.2)	$-.1878$ (-4.4)	.0198 (1.0)
Panama	.0447 (10.3)	.0338 (10.6)	$-.2124$ (-5.6)	.0182 (1.3)
Philippines	.0513 (16.1)	.0174 (8.2)	$-.0588$ (-9.9)	.1523 (2.4)

Table 1: **GDP Dynamics of Less Developed Countries.** The data set is generated from The World Economy, OECD Development Centre, Paris 2003, consisting of yearly observations between 1974 and 2003. Estimation is carried out via maximum likelihood, and the t-statistics are displayed in parenthesis below the point estimates.

Our equilibrium pricing exercise carries elements from both reduced-form as well as structural models of defaultable debt. The default decision is modelled in reduced form since an exogenously triggered rare event leads to default on the debt contract. From this perspective our paper relates to the family of models that treat default as an unpredictable sudden event, like Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1995), or Duffie and Singleton (1999). Our pricing model also relates to the family of structural models in which prices are derived from economic fundamentals. The first generation of structural models includes for example Merton (1974) and Black and Cox (1976). In these models a defaultable bond is a contingent claim on the borrower's assets, and interest rate and recovery rate are endogenously determined. The second generation of structural models includes for example Longstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001). Although these models allow for more complex capital structures and bond prices are determined by asset fundamentals, the recovery rate is an exogenous value and independent of the value of assets. Because of the endogeneity of the recovery rate, our pricing approach is similar in spirit to the first generation of structural models, in addition to being an equilibrium model.

The borrower in our pricing exercise is potentially a sovereign country. One strand of the literature explores the question how much the willingness to repay debt affect debt prices.

Sovereign debt literature dealing with the role of reputation and sanctions includes Eaton and Gersovitz (1981), as well as Bulow and Rogoff (1989a, 1989b). Gibson and Sundaresan (2001) provide a model based on trade sanctions to quantify the difference between corporate and sovereign yields spreads due to the absence of a bankruptcy code. In a more recent paper, Yue (2005) provides a dynamic model that captures endogenous default risk and endogenous recovery, also based on debt renegotiation. In our model, the borrower does not trade off debt payments against costs of reputation, costs of international trade restrictions, or costs of having assets seized that are held abroad. Our model also does not build up on the small open economy framework, which is based on a risk averse borrower and a risk neutral financial intermediary. Since our pricing exercise is a risk sharing question with respect to a systematic source of uncertainty, we support the setup of a risk averse borrower and a risk averse lender. At the heart of the equilibrium pricing is the generation of a risk premium implicit in the interest rate of the debt contract, on which the borrower and the lender need to agree.

In other theoretical work on sovereign debt, Claessens and Pennachi (1996) provide a structural model for the pricing of Brady bonds. An (unobserved) state variable serves as an indicator that governs the country's ability to repay. Subsequently, default is defined as a stopping time associated with this state variable. The authors apply the model to Mexican Brady bond prices between 1990 and 1995, and analyze properties of the extracted state variable. Duffie, Pedersen, and Singleton (2003) propose a reduced-form model within the class of affine term structure models to study Russian bond data between 1994 and 2000. The sample period includes Russia's default on sovereign debt in 1998. They show how prices can be obtained using a "default risk-adjusted" short rate model, and successfully estimate the model using maximum likelihood. Our model allows us to go a step further by endogenizing the expected loss given default as well as the default risk premium, and linking its structure to the process of economic fundamentals. In addition, we show how the stochastic nature of interest rates and credit spreads in non-default times can be an outcome of the combined effect of limited participation in the borrower's economy, and the dispersion of the borrower's and the lender's beliefs about the likelihood of default.

Our structural approach is further motivated by the empirical finding that measures of a less-developed country's ability to repay its outstanding debt seem to explain the level of borrowing rates and their dynamics. Edwards (1984) empirically measures the significance of several variables to explain the level of the sovereign debt yield spread of 19 less-developed countries between 1976 and 1980. He shows that the debt-output ratio (as measured by the ratio of total debt to GNP), an indicator for the degree of solvency, is a significant explanatory variable. In a similar study, Boehmer and Megginson (1990) investigate empirically whether liquidity or solvency factors can explain the price dynamics for syndicated loans of 12 less developed countries between 1985 and 1988. They find that the country's ability to repay outstanding debt (as measured by the ratio of total long-term debt to GNP and by the ratio of long-term debt to total exports) significantly explains the changes in secondary market prices. Ming (1998) performs a similar analysis on emerging market bond spreads, and confirms the importance of solvency variables. All of these empirical studies directly support the pricing approach taken in our paper, as we are modeling a solvency variable endogenously, which in turn determines the level and variability of the interest rate.

Empirical papers dealing with rare events in equity markets include early work by Jorion (1988), as well as more recent analysis by Eraker (2004), and Das and Uppal (2004). Johannes (2004) explicitly analyzes jumps in interest rate markets, and Piazzesi (2005) provides a term structure model which integrates Federal Reserve actions into bond prices. With rare events, the problem of contingent claim pricing is first considered by Merton (1976); the problem of portfolio selection is considered, for example by Merton (1971), Aase (1984), and Liu, Longstaff and Pan (2003). For an equilibrium treatment, Rietz (1988) adds a catastrophic state to the Arrow-Debreu exchange economy. Naik and Lee (1990) study an equilibrium model with rare events in order to price European options. Back (1991) studies a broad class of processes for which rare event risk premiums exist, and shows how a non-zero jump risk premium is linked to jumps in the pricing kernel. More recently, Barro (2005) extends Rietz's (1988) framework with a default probability and addresses the equity premium puzzle and real interest rates puzzle in developed nations in the last century. While Barro (2005) assumes homogeneity and a complete markets economy, our setup is based on limited participation through a debt contract and we endogenize the recovery rate.[7]

Our paper is structured a follows. Section 2 describes the economy under consideration and provides the pricing results of the debt contract, i.e. the interest rate, the write-off, the credit spread, and the risk premium. In Section 3, we analyze the properties of the

[7]Dieckmann and Gallmeyer (2005) add heterogeneity in risk aversion, and study a capital market in which the more risk averse optimally insures the less risk averse agent against rare events. Dieckmann (2004) analyzes an exchange economy with rare event risk, while focusing on the difference between complete and incomplete capital markets and the non-availability of insurance.

debt contract and the dynamic behavior in default and non-default times. In Section 4 we calibrate the model to data from Ecuador, and Section 5 concludes. In the Appendix we show a perturbation of the model allowing for a small degree of foreign direct investment, with a re-calibration to the case of Ecuador.

2 The Economy

First, we introduce the primitives of the economy and formalize the debt contract. Second, after stating the optimization problem, we solve for the equilibrium price characteristics of the debt contract. Additionally, we derive benchmark economies without rare event risk and without heterogeneous beliefs about rare event risk, respectively.

Our model is a continuous time generalization of a Lucas (1978) pure exchange economy with two agents. The first agent ($i = b$) represents a sovereign country which has access to its aggregate output process. At the same time this country acts as a borrower in the sovereign debt market. The second agent ($i = l$) represents a lender in the sovereign debt market. Both agents ($i = b, l$) observe the realization of the country's aggregate output process with the following exogenous dynamics:

$$\frac{de(t)}{e(t-)} = \widehat{\mu}_e dt + \sigma_e dB(t) + \kappa_e dN(t, \lambda_i(t)), \quad (i = b, l). \tag{1}$$

The output process carries two sources of uncertainty, small (regular) economic risk and rare event risk, as often observed in the case of emerging market countries. To capture the former, the economy is subject to uncertainty that enters through a one-dimensional standard Brownian motion $B(t)$. The extent of small economic risks is given by the instantaneous volatility parameter σ_e, which is strictly positive, $\sigma_e > 0$. To capture rare event risk, the economy is subject to uncertainty that enters through a one-dimensional Poisson process $N(t)$ with intensity parameter λ.[8] The extent of rare event risk is given by the jump size parameter κ_e. We restrict the jump size to $\kappa_e \in (-1, 0)$ in order to induce negative jumps and

[8]The Brownian motion is defined on a probability space $\left(\Omega^B, F^B, \mathcal{P}^B\right)$. The Poisson process is defined on the probability space $\left(\Omega^N, F^N, \mathcal{P}^N\right)$. We define $(\Omega, F, \mathcal{P})$ as the product probability space and the filtration of the combined history as $\{F_t\} = F_t^B \times F_t^N$.

to ensure that the output process always remains positive. Heterogeneity in beliefs about the likelihood of rare events (leading to subsequent default) is captured by each agent's subjective belief about the true frequency, $\lambda_i(t)$.[9] We formulate a competitive equilibrium, in which none of the agents knows the true frequency, and both fully agree to disagree on each other's belief. The coefficient $\widehat{\mu}_e$ serves as the deterministic growth rate. Alternatively, we can write the output process under a compensated Poisson process as

$$\frac{de(t)}{e(t-)} = (\mu_{e,i}(t) - \kappa_e \lambda_i(t))dt + \sigma_e dB(t) + \kappa_e dN(t, \lambda_i(t)), \tag{2}$$

which implies $\widehat{\mu}_e = \mu_{e,i}(t) - \kappa_e \lambda_i(t)$. Under this notation, $\mu_{e,i}(t)$ serves as the instantaneous mean growth rate of the output process under agent i's belief.[10]

The rare event risk inherent in the borrower's economy is the source of uncertainty linked to the default behavior. The default decision is modeled in reduced form, since a sudden shock in economic fundamentals of magnitude κ_e leads to default in the debt market. However, we will show in the next section that this link between the occurrence of a rare event and default is needed to obtain an equilibrium solution. Riskless borrowing for a levered sovereign country facing rare event risk in addition to regular economic risk is

[9] One might argue that disagreement on the extent of a rare event is more realistic than disagreement on the frequency. We consider these here as equivalent frictions, as long as the support for jump sizes is finite. Suppose a rare event can have either small or large impact. After decomposing the jump size uncertainty into a sum of two Poisson events with constant (small and large) jump sizes, one can subsume the small jump into the more general Brownian motion risk. What is left is a Poisson process with a large jump size, and disagreement on the frequency of occurrence. In other words, disagreement on the extent of the impact of a rare event can be interpreted as disagreement on the frequency of large rare event risk, which is the rare event that matters for our analysis. For a more rigorous treatment of multiple jump sizes, see Dieckmann (2004) and Dieckmann and Gallmeyer (2005.)

[10] Since both agents observe the same process, agreement on the path of $e(t)$ requires the consistency condition

$$\widehat{\mu}_e = \mu_{e,i}(t) - \kappa_e \lambda_i(t), \quad (i = b, l).$$

The difference in mean growth rates can be expressed in terms of the difference in beliefs about default, and must hold at all points in time,

$$\mu_{e,b}(t) - \mu_{e,l}(t) = \kappa_e(\lambda_b(t) - \lambda_l(t)).$$

impossible, since financial wealth might become negative. We assume that smaller risks in economic fundamentals will not lead to default. Investment by the lender in the borrower's economy occurs through a debt contract with the price process,

$$\begin{aligned}\frac{dD(t)}{D(t-)} &= \widehat{\mu}_D(t)dt + \kappa_D(t)dN(t, \lambda_i(t)), \qquad D(0) > 0, \qquad (3)\\ &= (\mu_{D,i}(t) - \lambda_i(t)\kappa_D(t))dt + \kappa_D(t)dN(t, \lambda_i(t)).\end{aligned}$$

For simplicity, this is a non-dividend paying security and behaves like a defaultable money market account. The initial value $D(0)$ is given exogenously. A long position in $D(t)$ classifies a lender; a short position in $D(t)$ classifies a borrower. The annual interest rate of borrowing/lending is given by $\widehat{\mu}_D(t)$. Hence, the borrowing agent pays the amount $D(t-)\widehat{\mu}_D(t)dt$ to the lending agent every instant of time. The impact of default is captured by the write-off parameter $\kappa_D(t)$, such that $1 - \kappa_D(t)$ represents the recovery rate. The notions "recovery of market value" and "recovery of face value" are equivalent in our formulation. In the case of an extreme shock to the output process triggered by $dN(t, \lambda)$, it is optimal to write-off the sovereign debt contract by the amount $D(t-)\kappa_D(t)$. The interest rate $\widehat{\mu}_D(t)$, and the write-off parameter $\kappa_D(t)$ are posited to be $\{F_t\}$ measurable, and to be determined jointly in equilibrium. In a complete market frictionless economy, the write-off parameter $\kappa_D(t)$ would not be determined in equilibrium, since $D(t)$ is a non-dividend paying security. However, in our incomplete market economy with limited market participation and an exogenous initial amount of total debt, the write-off parameter $\kappa_D(t)$ can be uniquely determined.

Agents have the following endowment structure at the beginning of the economy. The borrower is endowed with the present value of the entire output stream, denoted $V(0)$. Simultaneously, the borrower has issued $m_D(0)$ debt contracts, where $m_D(0) > 0$. Hence, the borrower's initial financial wealth is given by $W_b(0) = V(0) - m_D(0)D(0)$, which leads to the restriction that the initial amount of borrowing is limited by $m_D(0)D(0) < V(0)$. The lender's initial financial wealth is $W_l(0) = m_D(0)D(0)$. The amount $m_D(0)D(0)$ can also be interpreted as the initial amount of "leverage" for the borrower. In equilibrium, the financial wealth of both agents remains strictly positive at all times. In our formulation, the investor is constrained to participate in the borrower's economy through the sovereign debt contract. This is in line with very low levels of foreign direct investment observed for less developed countries.[11] To formalize this, let agent i's fraction of wealth invested in asset j

[11] For example, according to World Bank statistics, the average FDI in Ecuador between 1970 and 2004

be denoted by $\pi_{j.i}$, where $j \in \{D, V\}$ can either be the debt contract $D(t)$ or the value of the "unlevered" economy $V(t)$. The participation constraint corresponds to $\pi_{V,b}(t)W_b(t) = V(t)$, $\pi_{V,l}(t) = 0$, and $\pi_{D,l}(t) = 1$ at all points in time.

We propose the following dynamics for the process for an agent-specific pricing kernel to be verified in equilibrium:

$$\begin{aligned} \frac{d\eta_i(t)}{\eta_i(t-)} &= -\left(\widehat{\mu}_D(t) + \alpha(t) - \lambda_i(t) + \lambda_{Q,i}(t)\right) dt - \theta_i(t) dB(t) \\ &\quad + \left(\frac{\lambda_{Q,i}(t) - \kappa_D(t)\lambda_i(t)}{\lambda_i(t)} - 1\right) dN(t, \lambda_i(t)). \end{aligned} \tag{4}$$

The parameters θ_i and $\lambda_{Q,i}$ serve as the market price of diffusive risk associated with the Brownian motion, and the risk-adjusted jump intensity associated with the Poisson process, respectively. Note, $\lambda_{Q,i}$ is the risk-adjusted parameter, and not the risk neutral equivalent. The proposed capital market is incomplete since the only security available to share risk is the debt contract. We verify later that in equilibrium the coefficient $\alpha(t)$ is equal to the term $-\kappa_D^2(t)\lambda_i - \kappa_D(t)\lambda_i + \lambda_{Q,i}(t)\kappa_D(t)$. Note that $\alpha(t)$ does not carry a subscript i, as the value is not agent-specific. Furthermore, the expectation of $\frac{d\eta_i(t)}{\eta_i(t-)}$ is equal to

$$\begin{aligned} E_{i,t}\left[\frac{d\eta_i(t)}{\eta_i(t-)}\right] &= [-\widehat{\mu}_D(t) - \alpha(t) - \kappa_D(t)\lambda_i(t)]dt \\ &= [-\widehat{\mu}_D(t) + \kappa_D(t)(\kappa_D(t)\lambda_i(t) - \lambda_{Q,i}(t))]dt. \end{aligned} \tag{5}$$

One can interpret the right hand side of equation 5 as the "shadow" riskless rate. This value is agent-specific, and there would is disagreement about it between agents. If there is no rare event risk in the economy, $\kappa_e(t) = 0$, then the right hand side of equation 5 yields $-r(t)dt$, with full agreement on the riskless rate.

The borrower and the lender solve an optimization problem over a finite horizon in an expected utility framework by being endowed with logarithmic utility. They choose a nonnegative consumption process c_i. Using martingale techniques, see Karatzas, Lehoczky

was 2% of GDP. In the Appendix we relax the assumption of zero FDI, and provide the new debt contract properties for a small amount of FDI.

and Shreve (1987), Cox and Huang (1989) as well as Bardhan and Chao (1996),[12] a static version of each agent's optimization problem is given by

$$\max_{c_i} E_i \left[\int_0^T \log(c_i(t))dt \right] \;\; s.t. \; E_i \left[\int_0^T \eta_i(t)c_i(t)dt \right] = \eta_i(0)W_i(0). \tag{6}$$

The optimal consumption policy can be determined from the inverse of each agent's marginal utility and yields $c_i(t) = (y_i\eta_i(t))^{-1}$. The parameter y_i serves as the Lagrangian multiplier from agent i's constrained optimization. After solving for the value of y_i, financial wealth simplifies to $W_i(t) = c_i(t)(T-t)$. Its dynamics, while focusing on the terms generated by Brownian and Poisson uncertainty, relate to the inverse of the agent-specific state price density process η_i given by

$$\frac{dW_i(t)}{W_i(t-)} = (...)dt + \theta_i(t)dB(t) + \left(\frac{\lambda_i(t)}{\lambda_{Q,i}(t) - \kappa_D(t)\lambda_i(t)} - 1 \right) dN(t, \lambda_i(t)). \tag{7}$$

Definition 1 *Given both agent's preferences and endowments, a Walrasian equilibrium is a collection of allocations* $(c_b, \pi_{j,b})$ *and* $(c_l, \pi_{j,l})$, *and a price system for the sovereign debt market* $(\widehat{\mu}_D(t), \kappa_D(t))$, *such that* $(c_b, \pi_{j,b})$ *and* $(c_l, \pi_{j,l})$ *are optimal solutions to the agent's optimization problem. All markets clear at* $t \in [0, T]$*:*

$$\begin{aligned}
c_b(t) + c_l(t) &= e(t), \\
W_b(t) + W_l(t) &= V(t), \\
\pi_{V,b}(t)W_b(t) &= V(t), \quad \textit{(borrower has access to the economy)} \\
\pi_{D,b}(t)W_b(t) + \pi_{D,l}(t)W_l(t) &= 0. \quad \textit{(debt market clearing)}
\end{aligned}$$

We now solve for the equilibrium of this economy, and find it convenient to construct a representative agent (RA) with a state-dependent weight $\phi(t)$. A RA utility function where the first weight is normalized to unity can be formulated as

$$U(e(t), \phi(t)) = \max \;\; \log(c_b(t)) + \phi(t)\log(c_l(t)), \;\; s.t. \; c_b(t) + c_l(t) = e(t).$$

As usual, optimality and consumption good clearing imply that the RA's marginal utility equates to first agent's state price density, i.e.

[12]Bardhan and Chao (1996) study an exchange economy with rare events where multiple agents can trade in a security market and also face individual endowment streams.

$$\eta_b(t) = \frac{U_c(e(t), \phi(t))}{U_c(e(0), \phi(0))}, \quad \eta_l(t) = \frac{\phi(0)}{\phi(t)}\eta_b(t). \tag{8}$$

The weighting process is given by

$$\phi(t) = \frac{y_b \eta_b(t)}{y_l \eta_l(t)}, \tag{9}$$

and the consumption sharing rule by

$$c_b(t) = \frac{e(t)}{1 + \phi(t)}, \qquad c_l(t) = \frac{\phi(t)e(t)}{1 + \phi(t)}.$$

The approach to formulate a RA with state-dependent weights was introduced by Cuoco and He (1994), and recent examples can be found in Basak and Cuoco (1998) or Gallmeyer and Hollifield (2004). The value of $\phi(t)$ needs to be strictly positive in order to guarantee existence of an equilibrium, which will be directly verified in equilibrium. A shortcut to determine the present value of the output process, $V(t)$, is to use the market clearing conditions. As $W_i(t) = c_i(t)(T - t)$, imposing good market clearing leads directly to $V(t) = e(t)(T - t)$. Hence, after computing the dynamics of $V(t)$, the instantaneous volatility and the jump size are the same values as for the output process, i.e. σ_e and κ_e, respectively.

The market prices of diffusive risk can be determined as in an equilibrium model in which one agent holds the entire exposure of diffusive risk, but the other agent stays entirely sidelined, as in Basak and Cuoco (1998). With this restricted risk sharing, it must be the case that $\theta_b(t) = \sigma_e(1 + \phi(t-))$ and $\theta_l(t) = 0$, and the dynamics of equation (9) reduce to

$$\begin{aligned} \frac{d\phi(t)}{\phi(t-)} &= (\lambda_b(t) - \lambda_{Q,b}(t) - \lambda_l(t) + \lambda_{Q,l}(t))dt - \theta_b(t)dB(t) \\ &\quad + \left(\frac{\lambda_l(t)\left(\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b(t)\right)}{\lambda_b(t)\left(\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l(t)\right)} - 1 \right) dN(t, \lambda_i(t)). \end{aligned} \tag{10}$$

Note that the dynamics of the weighting process do not depend on $\widehat{\mu}_D(t)$ and $\alpha(t)$. These coefficients are not agent-specific and therefore drop out of the weighting process. Our equilibrium pricing exercise involves determining the following four processes: the interest rate $\widehat{\mu}_D(t)$, the risk-adjusted intensity for the borrowing agent $\lambda_{Q,b}(t)$, the risk-adjusted intensity for the lender $\lambda_{Q,l}(t)$, and the write-off in case of default $\kappa_D(t)$. In the following, we state four equilibrium conditions, which allow us to uniquely identify the solution.

Condition 1: The borrower has access to the entire aggregate output process, and under her pricing kernel the value of the economy must equal $V(t)$. Since $V(t) = e(t)(T-t)$ as stated above, it must also follow for consistency that

$$V(t) = \frac{1}{\eta_b(t)} E_{b,t}\left[\int_t^T \eta_b(s)e(s)ds\right] = e(t)(T-t). \tag{11}$$

After imposing optimality, $\frac{1+\phi(t)}{e(t)y_b} = \eta_b(t)$, this condition requires that the weighting process is a martingale under the sovereign's measure $\mathcal{P}_b$, $E_{b,t}[\phi(s)] = \phi(t)$, for all $s > t$. Equivalently we require

$$\lambda_{Q,l}(t) - \lambda_{Q,b}(t) - \lambda_l(t) + \frac{\lambda_l(t)\left(\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b\right)}{\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l} = 0. \tag{12}$$

Condition 2: The lender is exposed to the entire risky debt contract. Hence, at the occurrence of default his percentage change in wealth, $\frac{dW_l(t)}{W_l(t-)}$, should equal the percentage write-off on the debt contract $\kappa_D(t)$. Intuitively, this is the condition that allows us to pin down the write-off parameter on the debt contract, although being a non-dividend paying security:

$$\left(\frac{\lambda_l(t)}{\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l(t)} - 1\right) = \kappa_D(t). \tag{13}$$

Condition 3: By construction the dynamics of the RA's marginal utility equate with the first agent's state price density process. The continuous part of the dynamics of the RA's marginal utility is given by the following equation. For this continuous component, $\phi(t)$ and $e(t)$ are equivalent to $\phi(t-)$ and $e(t-)$, and we can simplify to

$$\begin{aligned}\frac{d\left[\frac{1+\phi(t)}{e(t)}\right]^c}{\frac{1+\phi(t-)}{e(t-)}} &= \left[\frac{\phi(t-)}{1+\phi(t-)}[\lambda_b(t) - \lambda_{Q,b}(t) - \lambda_l(t) + \lambda_{Q,l}(t) + \theta_b(t)\sigma_e] - \widehat{\mu}_e + \sigma_e^2\right] dt \\ &\quad - \left[\frac{\phi(t-)}{1+\phi(t-)}\theta_b(t) + \sigma_e\right] dB(t). \end{aligned} \tag{14}$$

For the discontinuous part, the dynamics of the RA's marginal utility are given by

$$\frac{d\left[\frac{1+\phi(t)}{e(t)}\right]^{dc}}{\frac{1+\phi(t-)}{e(t-)}} = \left[\frac{\frac{1}{1+\phi(t-)} + \frac{\phi(t-)}{1+\phi(t-)}\left(\frac{\lambda_l(t)\left(\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b(t)\right)}{\lambda_b(t)\left(\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l(t)\right)}\right)}{(\kappa_e + 1)} - 1\right] dN(t, \lambda_i(t)). \tag{15}$$

Comparing the $dN(t)$ terms of equations (4) and (15) leads to the third condition:

$$\frac{\frac{1}{1+\phi(t-)}+\frac{\phi(t-)}{1+\phi(t-)}\frac{\lambda_l(t)\big(\lambda_{Q,b}(t)-\kappa_D(t)\lambda_b(t)\big)}{\lambda_b(t)\big(\lambda_{Q,l}(t)-\kappa_D(t)\lambda_l(t)\big)}}{(\kappa_e+1)}=\frac{\lambda_{Q,b}(t)-\kappa_D(t)\lambda_b(t)}{\lambda_b(t)}. \tag{16}$$

Condition 4: Comparing the deterministic terms of equations 4 and 14 leads to the fourth condition, and thereby to the identification of the interest rate $\mu_D(t)$.

These four conditions have a unique solution, as summarized in Proposition 1, and are the basis of our equilibrium characterization. Financial wealth is allocated either towards the claim on the output process $V(t)$ or the sovereign debt contract $D(t)$. It can easily be verified that the dynamics of the wealth process in equation (7), equate with the wealth dynamics generated from the process of $V(t)$ and $D(t)$,

$$\frac{dW_i(t)}{W_i(t-)}=(...)dt+\pi_{V,i}(t)\sigma_e dB(t)+(\pi_{V,i}(t)\kappa_e+\pi_{D,i}(t)\kappa_D(t))dN(t,\lambda_i(t)) \tag{17}$$

given the equilibrium allocations $\pi_{V,b}(t)=1+\phi(t-)$, $\pi_{V,l}(t)=0$, $\pi_{D,b}(t)=-\phi(t-)$, and $\pi_{D,l}(t)=1$.

Proposition 1 *The interest rate $\widehat{\mu}_D(t)$, the risk-adjusted intensity for the borrowing agent $\lambda_{Q,b}(t)$, the risk-adjusted intensity for the lender $\lambda_{Q,l}(t)$, and the write-off in case of default $\kappa_D(t)$, are given in closed form by*

$$\widehat{\mu}_D(t)=\widehat{\mu}_e-\sigma_e^2(1+\phi(t-))+\frac{1}{1+\phi(t-)}[\lambda_b(t)-\lambda_{Q,b}(t)]+\frac{\phi(t-)}{1+\phi(t-)}[\lambda_l(t)-\lambda_{Q,l}(t)]-\alpha(t), \tag{18}$$

$$\lambda_{Q,b}(t)=\frac{(\kappa_D(t)(-1+\phi(t-)\kappa_D(t)-(1+\phi(t-))\kappa_e)-1)\lambda_b(t)}{\phi(t-)\kappa_D(t)-(1+\phi(t-))\kappa_e-1}, \tag{19}$$

$$\lambda_{Q,l}(t)=\frac{(1+\kappa_D(t)+\kappa_D(t)^2)\lambda_l(t)}{1+\kappa_D(t)}, \tag{20}$$

$$\textit{If } \lambda_b(t)=\lambda_l(t),\ \textit{ then } \kappa_D(t)=\kappa_e, \tag{21}$$

otherwise

$$
\begin{aligned}
\kappa_D(t) &= \frac{1}{2\phi(t-)\overline{\lambda}(t)}\left[\beta(t)+\sqrt{4\phi(t-)(1+\phi(t-))\kappa_e\lambda_b(t)\overline{\lambda}(t)+(\beta(t))^2}\right], \qquad (22)\\
\textit{where } \beta(t) &= ((1+\phi(t-))\kappa_e)\lambda_b(t)-(1+(1+\phi(t-))\kappa_e-\phi(t-))\lambda_l(t),\\
\textit{and } \overline{\lambda}(t) &= \lambda_b(t)-\lambda_l(t).
\end{aligned}
$$

The interest rate in equation (18) carries an intuitive interpretation. First, if there was no uncertainty in the economy, lending and borrowing would obviously be riskless at the exogenous growth rate $\widehat{\mu}_e$ of the economy. Second, adding Brownian motion risk results in a precautionary savings term, $\sigma_e^2(1+\phi(t-))$, which takes into account that the first agent's consumption policy carries the Brownian motion risk of the entire dividend stream. Third, adding rare event risk adds two new terms, one for each agent. This is each agent's wealth-weighted contribution to the rare event premium in the economy, the difference between the physical and the risk-adjusted intensity. For the lender, the risk-adjusted frequency is strictly higher than the belief about the physical frequency. This is expected, as this difference is a measure of the lender's required risk premium. However, for the borrower, the difference between the belief about the physical frequency and risk-adjusted frequency can be negative. The effect can be so large that the borrower is willing to pay an interest rate to the lender, which is higher than the growth rate in fundamentals $\widehat{\mu}_e$. As mentioned above, the borrower and the lender determine a "shadow" risk free interest rate of the sovereign country, given by $r(t)=\widehat{\mu}_D(t)-\kappa_D(t)(\kappa_D(t)\lambda_i(t)-\lambda_{Q,i}(t))$. Consequently, each agent's (instantaneous) credit spread is given by

$$
v_i(t)=\kappa_D(t)(\kappa_D(t)\lambda_i(t)-\lambda_{Q,i}(t)). \qquad (23)
$$

By decomposing $v_i(t)$ into the expected loss and the default risk premium due to rare event risk, we notice that the expected loss obviously is an agent-specific term, but borrower and lender agree on the risk premium given by $v_i(t)+\kappa_D(t)\lambda_i(t)$. Finally, the term $\alpha(t)$ enters the drift of the state price density process, such that the only marketable security, the debt contract $D(t)$, satisfies the martingale property as stated in Proposition 2. Equivalently, this result says that the debt contract satisfies the Euler equation of asset pricing.

Proposition 2 *The deflated process for security $D(t)$ satisfies the martingale property under each agent's measure $\mathcal{P}_i$, such that $E_{i,t}[\eta_i(s)D(s)]=\eta_i(t)D(t)$, for all $s>t$.*

A well understood benchmark of this economy is the case in which there is no rare event risk, given by a zero jump size in the exogenous process, $\kappa_e = 0$. In this case, one agent has access to the entire dividend stream and can borrow through a riskless money market. The second agent engages as a lender in the money market, like in the case of limited participation as in Basak and Cuoco (1998).

Corollary 1 *As the jump in the exogenous jump size approaches zero, $\kappa_e \to 0$, the borrowing contract becomes locally riskless, i.e.*

$$\frac{dD(t)}{D(t-)} \to r(t)dt, \qquad D(0) > 0,$$

in which the interest rate is given by

$$\widehat{\mu}_D(t) = r(t) = \widehat{\mu}_e - \sigma_e^2(1 + \phi(t)),$$

and the dynamics of the weighting process become

$$\frac{d\phi(t)}{\phi(t)} = -\theta_b(t)dB(t),$$

as in Basak and Cuoco (1998). Furthermore, the equilibrium characteristics in Proposition 1 converge to $\lambda_{Q,i}(t) = \lambda_i(t)$, $\kappa_D(t) = 0$, $\alpha(t) = 0$, and the state price density process to

$$\frac{d\eta_i(t)}{\eta_i(t-)} = -r(t)dt - \theta_i(t)dB(t).$$

The second benchmark of this economy is the case in which there is rare event risk, but homogeneous beliefs about the likelihood as formalized in the following Corollary. Although $\widehat{\mu}_D(t)$ has the same functional form as without rare event risk, it can not be interpreted as a riskless interest rate. This only says that the adjustment for the precautionary savings motive as a response to small economic risk is the same as in an economy without rare event risk.

Corollary 2 *For the case in which $\lambda_b(t) = \lambda_l(t)$, the write-off in case of a rare event (default) is equal to the exogenous jump,*

$$\kappa_D(t) = \kappa_e,$$

agents agree on the risk-adjusted frequencies,

$$\lambda_{Q,b}(t) = \lambda_{Q,l}(t) = \frac{(1+\kappa_e+\kappa_e^2)\lambda_i(t)}{1+\kappa_e},$$

the interest rate simplifies to

$$\begin{aligned}\widehat{\mu}_D(t) &= \widehat{\mu}_e - \sigma_e^2(1+\phi(t-)) + \lambda_i(t) - \lambda_{Q,i}(t) - \alpha(t) \\ &= \widehat{\mu}_e - \sigma_e^2(1+\phi(t-)),\end{aligned}$$

and the dynamics of the weighting process become

$$\frac{d\phi(t)}{\phi(t)} = -\theta_b(t)dB(t).$$

Before moving on to the results discussion, we emphasize some features of the agent-specific pricing operator, $\eta_i(t)$. It satisfies good market clearing, $c_b(t) + c_l(t) = (y_b\eta_b(t))^{-1} + (y_l\eta_l(t))^{-1} = e(t)$, and the deflated price process of the only tradable security, the sovereign debt contract, is a martingale under each agent's belief, $E_{i,t}[\eta_i(s)D(s)] = \eta_i(t)D(t)$. Most important for our particular problem, the choice of $\eta_i(t)$ allows us to determine the size of the write-off, $\kappa_D(t)$, which is the jump size of a non-dividend paying security, in equilibrium. One can show that the dynamics of $\eta_i(t)$ are well defined, the jump size $\left(\frac{\lambda_{Q,i}(t)-\kappa_D(t)\lambda_i}{\lambda_i(t)} - 1\right)$ is strictly bounded below at -1, and the risk-adjusted frequencies $\lambda_{Q,i}(t)$ are strictly positive.

3 Analysis and Results

The first thing to learn from this pricing exercise is that the borrower's debt contract cannot be a riskless contract. It is always optimal for the borrower to shift some exposure generated from rare event risk to the lender, i.e. default on the debt contract. Intuitively this is not a surprising result, as the debt contract is the financial instrument that allows the levered borrower to hedge against the possibility of negative financial wealth. This result is formalized in Proposition 3.[13]

[13]This Proposition has an equivalent interpretation in the context of the benchmark setup without rare event risk as formulated by Basak and Cuoco (1989). It says than an equilibrium solution based on a jump-diffusive process instead of a pure diffusive process under the same set of financial securities as in their paper

Proposition 3 *The debt contract is a risky security with non-zero default risk, such that*

$$-1 < \kappa_D(t) < 0, \tag{24}$$

for all $\phi(t) > 0, \lambda_b(t) > 0, \lambda_l(t) > 0$.

The write-off of the debt contract implements the risk sharing rule for rare event risk, conditioning on how frequent the borrower and the lender believe these will occur. The output process of the borrower faces a negative shock of magnitude κ_e. Consequently both borrower and lender lose financial wealth in case of a rare event (default). Proposition 4 studies how heterogeneity in beliefs about the rare event affects each agent's wealth in relative terms. Ex-ante, the agent who has a higher belief about the likelihood is willing to accept less exposure. This leads to the interesting result that although both agents lose in terms of financial wealth, the more "cautious" agent gains in relative terms, compared to the less cautious agent at the occurrence of default.

Proposition 4 *Each agent's exposure with respect to rare event risk as a fraction of financial wealth is given by*

$$\kappa_{W_i}(t) = \frac{W_i(t) - W_i(t-)}{W_i(t-)} = \left(\frac{\lambda_i(t)}{\lambda_{Q,i}(t) - \kappa_D(t)\lambda_i(t)} - 1\right).$$

The lending agent loses the fraction $\kappa_D(t) = \kappa_{W_l}(t)$*, the borrowing agent loses the fraction* $(1 + \phi(t))\kappa_e - \phi(t)\kappa_D(t) = \kappa_{W_b}(t)$*. Furthermore,*

$$\begin{aligned}
\text{If } \lambda_b(t) &= \lambda_l(t), \text{ then } \kappa_{W_b}(t) = \kappa_{W_l}(t) < 0, \\
\text{if } \lambda_b(t) &> \lambda_l(t), \text{ then } 0 > \kappa_{W_b}(t) > \kappa_{W_l}(t), \\
\text{if } \lambda_b(t) &< \lambda_l(t), \text{ then } \kappa_{W_b}(t) < \kappa_{W_l}(t) < 0.
\end{aligned}$$

At the occurrence of default, if the borrower and the lender agree on the likelihood, then relative wealth does not change; if $\lambda_b(t) > \lambda_l(t)$ *then the borrower gains relative to the lender, if* $\lambda_b(t) < \lambda_l(t)$ *then the borrower loses relative to the lender in terms of financial wealth.*

cannot be obtained. As mentioned in the introduction, the intuition for this result can also be derived from a portfolio selection perspective. Once a price systems faces rare events, a levered portfolio optimizer will not be able to quickly reverse his portfolio in order to avoid negative wealth. Liu, Longstaff and Pan (2003) show that the only way to hedge this inability not to continuously control the portfolio is to not engage in a levered position.

3.1 Interest Rate and Write-Off

The dependencies of $\widehat{\mu}_D(t)$ on the beliefs about the likelihood of a rare event (default) are shown with a numerical example. In line with the point estimates for emerging market countries in the introduction, we impose the following parameters for model primitives. The deterministic growth rate of the economy is assumed to be 6%, with an annualized volatility of 6%, and a possible extreme shock to economic fundamentals with a magnitude of -20%. The less developed country has borrowed 50% in terms of value of its economy from a lender, leading to a wealth ratio $\phi(t-)$ of 1. For ease of interpretation, we express our results based on annualized default probabilities, dp_i, instead of Poisson frequencies. The annual default probability given by a Poisson distribution with instantaneous intensity λ_i is given by $\mathrm{dp}_i = 1 - \exp(-\lambda_i)$. The left (right) graph in Figure 1 shows the level of the interest rate (write-off parameter), while the subjective beliefs about the annual likelihood of default borrower and lender vary between 20% and 60%. While these values seems high in absolute terms, they are in line with a range of default probabilities extracted from bond prices issued by Brazil, Ecuador and Venezuela (see Narag (2004)).[14]

Figure 1 contains a surprising result on each dimension. First, the interest rate is increasing in the belief of the borrowing country. This increase can be dramatic in that the borrowing agent pays a higher rate to the lender than the growth rate of the economy – in our example up to a level of 16%. This is possible as the lender is able to write-off more than the actual negative shock in the economic fundamentals in case of default, as shown in the right graph of Figure 1. This result is surprising, since the well-studied endowment economies based on homogeneity of beliefs, or an economy without rare event risk would generate interest rates strictly lower than the growth rate, see Corollaries 1 and 2. Second, the interest rate is decreasing in the belief of the lending agent. In a reduced form model in which the write-off is assumed to be constant, a higher anticipation of the probability of default would lead to a higher interest rate on the debt contract. Again, our result stems from the endogeneity of the write-off parameter. The write-off is less negative for higher values of the

[14] Narag (2004) extracts risk-neutral default probabilities from emerging market bond prices between 1997 and 2001, based on two reduced-form models. Since those are risk neutral probabilities, the corresponding physical default probabilities will be strictly lower in an environment where default risk is systematic, and economic agents are risk averse.

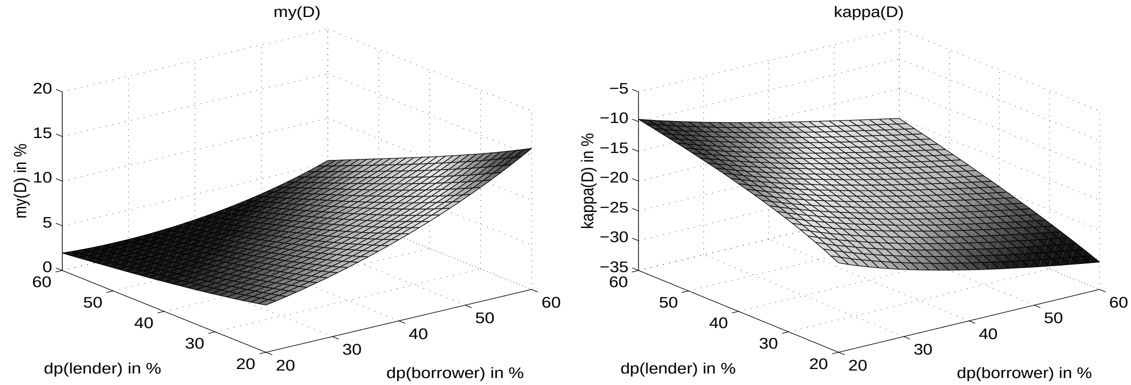

Figure 1: **Interest rate and write-off parameter.** The left graph shows the level of the interest rate $\mu_D(t)$ in %, the right graph shows the level of the write-off $\kappa_D(t)$ in %, as a function of the likelihood of default. The borrower's belief, dp(borrower), is shown on the x-axis, the lender's belief, dp(lender), is shown on the y-axis, between the values of 20% and 60%. Other parameters for the economy are $\widehat{\mu}_e = .06$, $\sigma_e = .06, \kappa_e = -.20, \phi(t-) = 1$.

lending agent's belief. In equilibrium, the lending agent expects a higher recovery rate the higher her anticipated degree of default risk. A higher recovery rate overcompensates higher anticipated default risk leading to a lower interest rate.[15]

For homogenous beliefs (the 45 degree line on the x-y dimension) neither the interest rate, nor the level of write-off depend on the frequencies of any agent. While this seems counterintuitive, it is consistent with standard models based on logarithmic preferences, and already documented in Corollary 2. For homogeneous beliefs, the interest rate is also independent of the jump size κ_e. If $\text{dp}_b < \text{dp}_l$, then the more negative κ_e, the lower the interest rate; if $\text{dp}_b > \text{dp}_l$, then the more negative κ_e, the higher the interest rate. We find the well-known comparative statics for the interest rate with respect to the deterministic growth rate, the instantaneous volatility and the jump size. For any given weight $\phi(t)$, the higher the growth rate $\widehat{\mu}_e$, the higher the rate of borrowing. The higher the degree of instantaneous volatility of small risks, σ_e, the lower the rate of borrowing. These results are equivalent to an economy without default risk, see Corollary 1.

[15] In addition to the short term debt contract $D(t)$, we have also computed long-term term structures based on the equilibrium pricing kernels $\eta_i(t)$, and the equilibrium write-off $\kappa_D(t)$ in case of default. The general finding is our model can produce upward-sloping as well as downward-sloping term structures, depending on the dispersion of beliefs and the initial degree borrowing and lending.

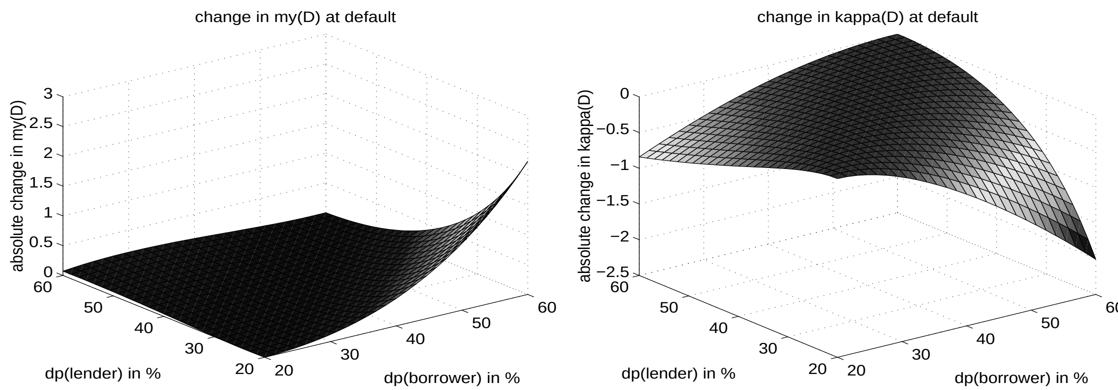

Figure 2: **Adjustment of the interest rate and the write-off in default times.** The left graph shows the absolute change in the interest rate $\mu_D(t)$, the right graph shows the absolute change in the write-off $\kappa_D(t)$, as a function of the likelihood of default. The borrower's belief, dp(borrower), is shown on the x-axis, the lender's belief, dp(lender), is shown on the y-axis, between the values of 20% and 60%. Other parameters for the economy are $\widehat{\mu}_e = .06$, $\sigma_e = .06, \kappa_e = -.20, \phi(t-) = 1$.

Since this is a dynamic model, we can ask how the interest rate and the write-off parameter adjust at a default. The right graph of Figure 1 already provides the necessary information to answer that question. Consistent with Proposition 4, if $\text{dp}_b < \text{dp}_l$, then κ_e is more negative than the write-off $\kappa_D(t)$. The lender gains in terms of wealth relative to the borrower, and $\phi(t-)$ jumps to a value higher at the occurrence of default. If $\text{dp}_b > \text{dp}_l$, then κ_e is less negative than the write-off $\kappa_D(t)$. In this case, the borrower gains in terms of wealth relative to the lender, and $\phi(t-)$ drops to a value smaller at the occurrence of default.

The left graph of Figure 2 shows the adjustment of the interest rate in levels. A large effect can be observed for the case in which $\text{dp}_b > \text{dp}_l$. While the belief of the borrower is high at 60%, and the belief of the lender is low at 20%, the interest rate would jump upwards about 2% at the occurrence of default. In this case, the lender has lost financial wealth relative to the borrower, so the borrower's belief matters more leading to a higher compensation in the interest rate. For the region $\text{dp}_b < \text{dp}_l$ the effect is also strictly positive, but smaller in levels. The write-off parameter unambiguously jumps downward at the occurrence of default as can be seen in the right graph of Figure 2. For example, while the belief of the borrower is high at 60%, and the belief of the lender is low at 20%, the write-off would jump down about

-2%. In this case, the increased interest rate goes hand in hand with a higher write-off. For homogenous beliefs there is no change, as agents lose wealth equally in relative terms, again in line with the results in Proposition 4.[16]

An important result of our study stems from the variability of the interest rate and write-off in non-default times, as regular (small) economic risk of the borrower generates uncertainty in equilibrium properties. Due to the nonlinear structure of $\widehat{\mu}_D(t)$, we compute the degree of variability numerically. Interest rate variability in non-default times is given by the instantaneous volatility of $d\widehat{\mu}_D(t)$, solely generated through the endogenous state variable $\phi(t)$,

$$\begin{aligned} d\widehat{\mu}_D(t) &= \frac{\partial \widehat{\mu}_D(t)}{\partial \phi(t-)} d\phi(t)^c = [...]\, dt + \sigma_{\widehat{\mu}_D}(t) dB(t), \\ \text{where } \sigma_{\widehat{\mu}_D}(t) &= \frac{\partial \widehat{\mu}_D(t)}{\partial \phi(t-)} (-\sigma_e)(1+\phi(t-))\phi(t-). \end{aligned} \tag{25}$$

The instantaneous volatility of the write-off parameter is computed equivalently. As typical in an exchange economy like ours, interest rates can become zero or negative, and the instantaneous volatility of percentage changes $\frac{\sigma_{\widehat{\mu}_D}(t)}{\widehat{\mu}_D(t)}$ is ill-defined at $\widehat{\mu}_D(t) = 0$. However, the level of the interest rate for our numerical example is not close to reaching the value zero, and for reasons of interpretation we decide to show the more intuitive measure $\frac{\sigma_{\widehat{\mu}_D}(t)}{\widehat{\mu}_D(t)}$, the instantaneous volatility of percentage changes, in Figure 3.

The interest rate contains a small degree of volatility in the case of homogeneity. This is in line with an economy without any rare event risk and limited participation, see Corollaries 1 and 2. Most important, heterogeneity matters dramatically for the degree of volatility. In our example, the maximum volatility occurs for low default probabilities of the lender, and high default probabilities of the borrower. In this case, the volatility of the interest rate increases to 6%, which is several times higher than the value under homogeneity, or no default. This result stems from the joint effect of limited participation and heterogenous beliefs. We also observe that the direction of volatility can change due to a change in sign of $\frac{\partial \widehat{\mu}_D(t)}{\partial \phi(t-)}$. In general,

[16] The adjustments described here are entirely due to a wealth effect, as we assume that the borrower's and the lender's subjective beliefs do not change at default. A possible effect of (Bayesian) learning and/or a change in the true frequency would add an additional layer to this analysis. Although we do not generate a hedging demand against updating beliefs with logarithmic preferences, a revision of the level $\lambda_b(t)$ and $\lambda_l(t)$ would impact equilibrium quantities in non-default and default times.

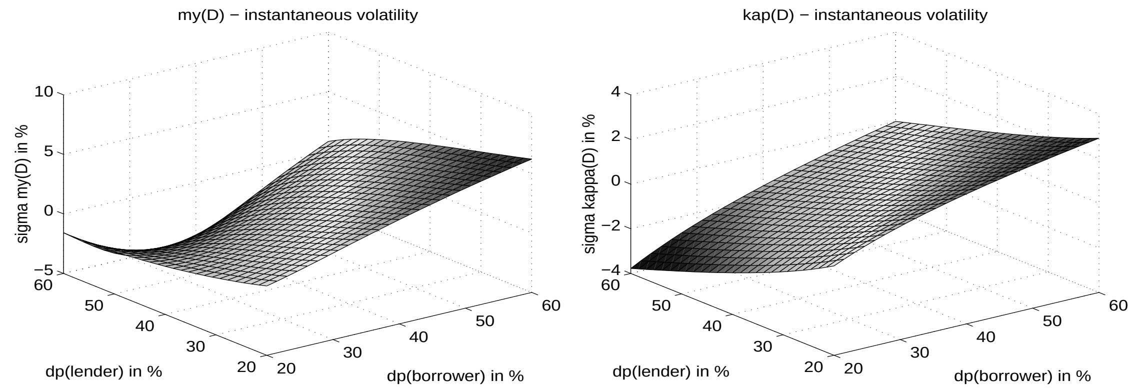

Figure 3: **Instantaneous volatility of the interest rate and the write-off in non-default times.** The left graph shows the level of volatility of percentage changes in the interest rate $\sigma_{\widehat{\mu}_D}(t)$ in %, the right graph shows the level of the volatility of percentage changes in the write-off parameter, as a function of the likelihood of default. The borrower's belief, dp(borrower), is shown on the x-axis, the lender's belief, dp(lender), is shown on the y-axis, between the values of 20% and 60%. Other parameters for the economy are $\widehat{\mu}_e = .06$, $\sigma_e = .06, \kappa_e = -.20, \phi(t-) = 1$.

a small positive shock in the output process makes the borrower wealthier relative to the lender due to the "levered" position, equivalent to a negative innovation in $\phi(t)$. However, this can have offsetting effects for the variability of the interest rate. The interest rate itself consists of multiple components — the riskless interest rate and a credit spread consisting of the expected loss and a default risk premium. We investigate these components in more detail in terms of level and variability in the next section. For completeness, the volatility of the write-off is displayed in the right graph of Figure 3.

3.2 Credit Spreads and Default Risk Premium

The credit spread, $v_i(t)$, implicit in the interest rate is agent-specific as the borrower and the lender face different "shadow" riskless interest rates. We find agreement between agents on the degree of the default risk premium. Figure 4 shows the level of the credit spreads, the default risk premium, as well as the instantaneous volatility in non-default times for the same parameters. The instantaneous volatility of the credit spread is given by

$$
\begin{aligned}
dv_i(t) &= \frac{\partial v_i(t)}{\partial \phi(t-)} d\phi(t)^c = [...] \, dt + \sigma_{v_i}(t) dB(t) \qquad (26) \\
\text{where } \sigma_{v_i}(t) &= \frac{\partial v_i(t)}{\partial \phi(t-)} (-\sigma_e)(1+\phi(t-))\phi(t-),
\end{aligned}
$$

and since all levels are bounded below at zero we present results for $\frac{\sigma_{v_i}(t)}{v_i(t)}$, the instantaneous volatility of percentage changes. The credit spread from the borrower's perspective is increasing in her own belief, but decreasing in the lender's belief, in line with the dependencies of the interest rate itself. However, the credit spread from the lender's perspective is increasing in her own belief, and increasing in the borrower's belief. We notice the large magnitude of the credit spreads — upto 32% from the borrower's perspective and upto 23% from the lender's perspective — and the level of the real riskless interest rate can be negative.

The risk premium contained in both of the spreads is consistently increasing in the belief of the borrower, but ambiguous in the belief of the lender. Interestingly, for low levels of the borrower's belief (as can been seen for a default probability of 20%), the risk premium is decreasing in the belief of the lender. This equilibrium effect is linked to the endogeneity of the write-off parameter. Recalling from Figure 1, the write-off parameter becomes less negative in this region, almost approaching a riskless asset, and in turn lowers the endogenous risk premium.

As before, the two main ingredients of the model result in a combined effect and lead to significant results regarding variability. Under homogeneity in beliefs, we do not observe any variation in credit spreads and the risk premium. Hence, the (small) variation in the interest rate observed earlier even under homogeneity is entirely due to variation in the "shadow" riskless interest rate. However, under heterogeneity in beliefs, small shocks to the borrower's output process generate large variation in credit spreads and the risk premium. The degree of variability is determined by the degree of limited participation in the borrower's economy, and the direction is determined by the sign of $\frac{\partial v_i(t)}{\partial \phi(t-)}$. In our numerical example, if the borrower's default probability is higher than the lender's probability then the risk premium's volatility can reach 7%; if the lender's default probability is higher than the borrower's probability then the volatility can reach almost 8%.

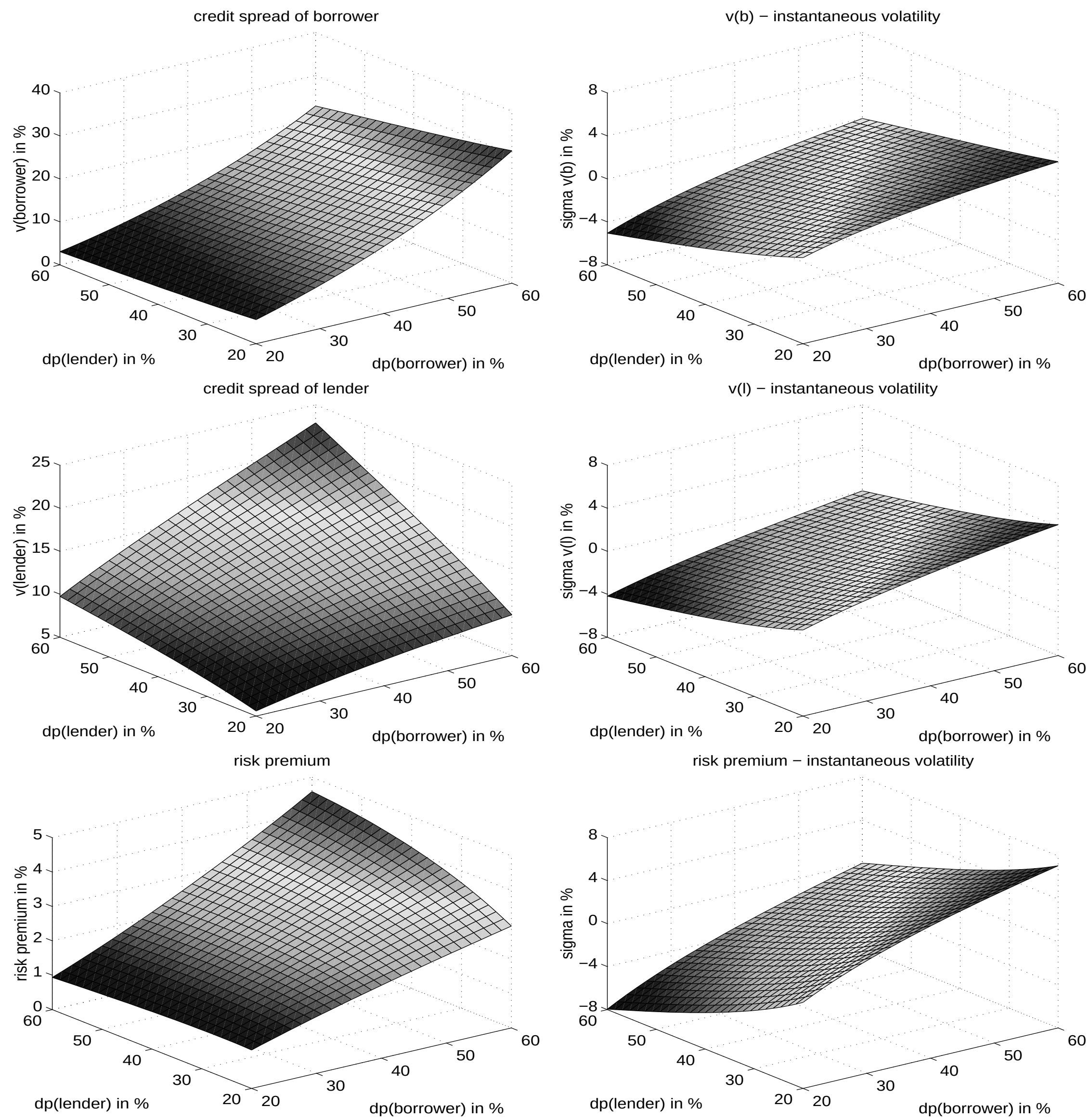

Figure 4: **Level and instantaneous volatility of the credit spread and the risk premium in non-default times.** The left graphs show the borrower's credit spread, the lender's credit spread, and the default risk premium in levels. The right graphs show the volatility of percentage changes of the same properties. Results are presented as a function of the likelihood of default. The borrower's belief, dp(borrower), is shown on the x-axis, the lender's belief, dp(lender), is shown on the y-axis, between the values of 20% and 60%. Other parameters for the economy are $\widehat{\mu}_e = .06$, $\sigma_e = .06, \kappa_e = -.20, \phi(t-) = 1$.

4 Calibration

The country of Ecuador is an example where severe shocks to economic fundamentals led to an enormous burden to repay interest and principal on external debt. The country's GDP suffered from the El Nino weather conditions, and further decreased significantly in 1998 and 1999 due to decreasing commodity prices and weaker export revenues. According to Deutsche Bank Research, the GDP decreased from 24 to 17 billion U.S. dollars between 1997 and 1999, with the most dramatic drop of about -20% between 1998 and 1999. According to Worldbank statistics, the effect was a total shock of -24% measured in terms of per capita GDP. Hence we consider $\kappa_e = -0.24$. In non-default times, the Worldbank per capita GDP time series between 1980 and 2002 had an annual standard deviation of 6.25% and a mean growth rate of 2% per annum. Hence we choose $\sigma_e = 0.0625$ and $\widehat{\mu}_e = 0.02$.

The value of external debt had increased to almost 16.3 billion U.S. dollar in 1999, representing 98% of GDP, and the fiscal deficit had risen to -4.6% of GDP. Ecuador first did not pay interest on one of its Brady bonds on August 28, 1999, and then officially suspended payment on half of the interest to be paid on its Brady bonds on October 1, 1999. Almost 6.1 billion U.S. dollars of external debt in 1999 were held in securitized Brady bonds or Eurobonds. In particular, Ecuador had issued 4 Brady bonds in 1995 with a face value of 5.6 billion U.S. dollars, and two Eurobonds with a total face value of 500 million U.S. dollars. Default on these bonds was resolved through a successful exchange offer made on July 27, 2000. The values received through the exchange offer can be interpreted as the recovery values on the 6 bonds. Brady bondholders received between 41% and 55% recovery of face value; Eurobond holders received 53% recovery of face value, according to Hund and Kulesz (2004.)

We choose the year prior to the impact of El Nino, 1997, as the basis for our calibration. A parameter to be calibrated is the degree of leverage, $\phi(t)$, or the ratio of Ecuador's financial wealth held by the lender relative to the borrower. The value of Ecuador's external debt in 1997 was 15.6 billion U.S. dollars. A potential proxy for wealth held by the borrower net of debt is Ecuador's stock market capitalization, which was 2.13 billion U.S. dollars as of 1997. While this would lead to a ratio of 7.32, one would also have to consider other sources of financial wealth not accounted for in the value of Ecuador's two stock exchanges. Hence, we calibrate our model to slightly more conservative values, and decide to show results for $\phi(t) = 6$, $\phi(t) = 4$, and $\phi(t) = 2$.

Interest rates generated in an exchange economy model like ours are real interest rates. Information on the time series of real interest rates from emerging market countries is very limited, and we calibrate our model to observed credit spreads, a measure potentially less impacted by inflation. A monthly data set of historical interest rates since 1995 is reported by the Bank of Ecuador (and obtained through Datastream). It contains a short term interest rate paid to a lender offering funds to Ecuador, based on U.S. dollar currency. We subtract the U.S. Treasury Bill rate, and consider the difference as the observed credit spread from the perspective of the lender. The descriptive statistics for the time period between 1995 and 1997 are a mean credit spread of 7.62%, a minimum of 5.75%, a maximum of 11.57%, and an annual standard deviation of percentage changes of 23%.

Along the lines of our theoretical study, we ask three question to be answered by our calibration. The borrower's and lender's belief about default are not directly observable. Hence, we first extract the pairs of (dp_b,dp_l) that would generate a credit spread of 7.62%. Second, we ask what debt write-off levels are implied by the answer to the first question, in order to draw inference about recovery rates from a pure risk sharing perspective. Third, we ask what levels of volatility of credit spreads in pre-default times can be explained by heterogeneity in beliefs, given Ecuador's highly levered economy.

Our calibration results are summarized in the three graphs in Figure 5. The upper graph shows the pairs of probabilities that generate a credit spread of 7.62%. A probability of approximately 21% leads to this value under homogeneous beliefs. Interestingly, this is close to the range of values extracted by Narag (2004) from Ecuador debt prices in 1997 between 25% and 40%. Please note that the author's results are risk-neutral values. They can only serve as an upper bound to the real counterpart, used in our calibration, in the presence of systematic risk. Relaxing the assumption of homogeneity, a continuum of pairs of probabilities can be found to match one statistical moment in observed data. For example, for the leverage ratio of $\phi(t) = 6$, a lender's belief of 20% and a borrower's belief of 40%, or a lender's belief of 33% and a borrower's belief of 5% would have generated the same credit spread in equilibrium.

The middle graph in Figure 5 shows that the implied write-off values are fairly insensitive to the degree of leverage. More importantly, within the range of probabilities extracted from the first step, write-off values between -15% and -26% can be supported by a rare event with an impact of $\kappa_e = -0.24$. Given the primitives of our model, this result can be compared to

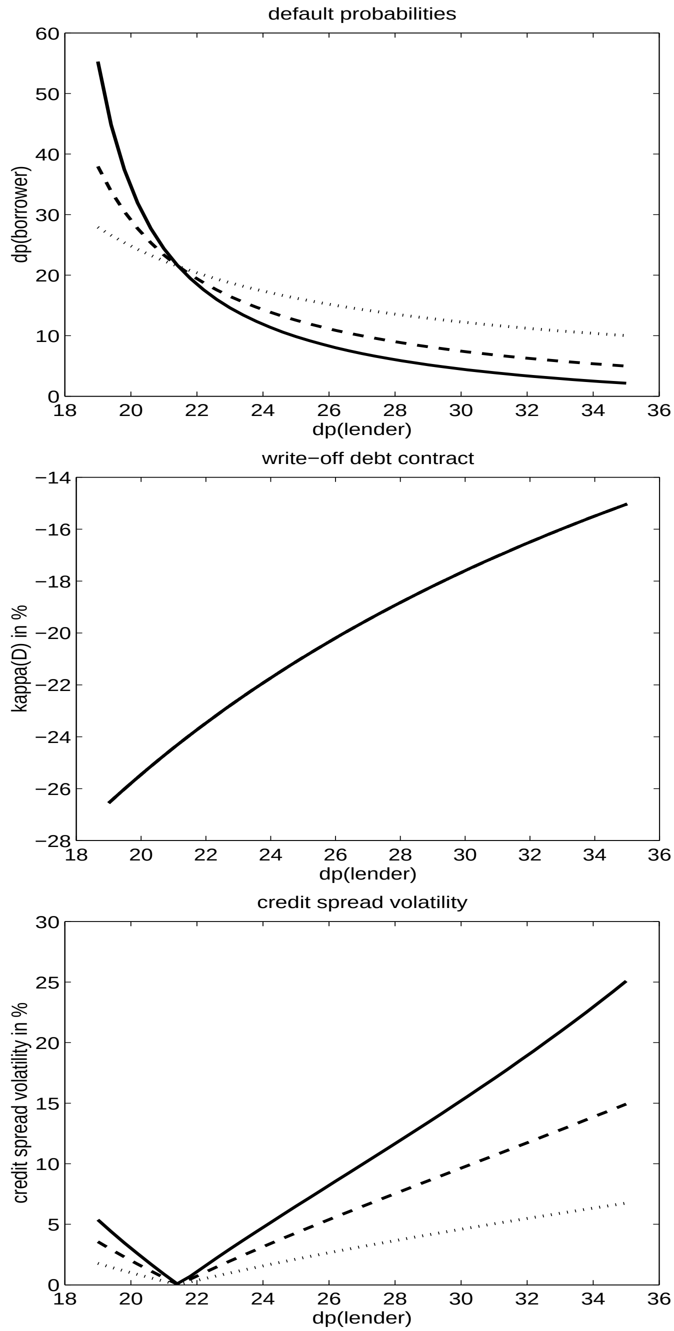

Figure 5: **Calibration results -** The upper graph shows the pairs of $(\mathrm{dp}_b, \mathrm{dp}_l)$ that generate a credit spread of 7.62% given $\kappa_e = -.24$. The solid, dashed, and dotted lines correspond to $\phi(t) = 6$, 4, and 2, respectively. The middle graph shows write-off κ_D that correspond to the pairs of $(\mathrm{dp}_b, \mathrm{dp}_l)$ extracted in the first step. The lower graph shows the corresponding level of instantaneous volatility in credit spreads expressed in absolute values given $\sigma_e = .0625$.

observed recovery rates mentioned above. It suggests that recovery rates (write-off values) on external debt, interpreted from a risk sharing perspective, should have been higher (lower) in the case of the Ecuador default.

The lower graph shows the implied levels of volatilities of credit spreads in pre-default times, expressed in absolute values. One can see clearly how variability in credit spreads is positively associated with a higher degree of leverage. As expected, homogeneity in default probabilities of approximately 21% generates no uncertainty in credit spreads. However, for a realistic level of leverage in 1997, $\phi(t) = 6$, a lender's belief of 33% and a borrower's belief of 5% would generate 23% volatility in credit spreads, which is precisely the descriptive level observed in our data set. While we are not claiming that heterogeneity in beliefs should explain all observed volatility in credit spreads, our results do, however, suggest that heterogeneity in beliefs can explain a significant fraction of Ecuador's observed volatility.[17] For example, a more moderate degree of dispersion, a lender's belief of 26% and a borrower's belief of 10%, can explain 10% volatility in credit spreads. Due to joint determination, this would have implied a write-off of -20% on debt, in line with a pre-default credit spread of 7.62%.

5 Conclusion

In this paper, we study the equilibrium problem of pricing a debt contract in an emerging market when the sovereign borrower faces both rare event and regular economic risk. The equilibrium is studied in the context of two natural market imperfections. First, the lender faces limited participation in the sovereign's economy by being able to invest only through the debt contract. Second, the lender and the borrower have heterogeneous beliefs about the likelihood of a rare event that triggers default by the borrower. We solve for the rate of interest, the credit spread, the risk premium, the write-off in case of default, and the dynamics of the debt contract in non-default times. The combination of limited participation and heterogeneous beliefs is shown to have strong implications for these equilibrium quantities. Additionally, our analysis highlights the importance of jointly modeling the debt contracts interest rate and recovery rate in that they are both closely linked in equilibrium.

[17] Other potential sources to generate amplifying effects in volatility are learning and updating of beliefs, or uncertainty in inflation not captured in our calibration to nominal credit spreads.

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7 Appendix

7.1 The Economy with Small Foreign Direct Investment

In this appendix we analyze a perturbation of the original economy by allowing for a small foreign direct investment (FDI). This is a form of investment in which an investor does not participate in the borrower's economy indirectly through the debt contract, but directly by investing in the economy's production possibilities. The goal of this perturbation analysis is to understand how equilibrium properties adjust once we relax the original assumption of having zero FDI, or a limited participation constraint of the lender on small economic risk.

Our results in Proposition 1 are derived from four conditions presented in Section 2. To analyze the perturbation, we show how these four conditions adjust once we include a small FDI, and then compute the new equilibrium properties numerically. To do so, we retain a "closed" economy, and divert a small fraction of lender's investment directly into the endowment stream $V(t)$. By construction the fraction of the lender's wealth invested in $V(t)$ is now $\pi_{V,l}(t) = \epsilon$, previously $\pi_{V,l}(t) = 0$. With respect to the two sources of risk inherent in the economy, the lender now has direct exposure to Brownian motion risk, i.e. small economic risk not related to default. The lender's market price of Brownian motion risk supporting this allocation is given by $\theta_l(t) = \epsilon\sigma_e$. Although this perturbation does not instantly change the wealth ratio between the borrower and the lender $\phi(t)$, it's interpretation as a "leverage ratio" of the country becomes diluted. More general, the ratio $\phi(t)$ is now the fraction of wealth held either directly through FDI or indirectly through a debt contract, relative to the fraction held by the borrower.

Since the borrower and the lender together bear the entire amount of risk, we can determine the borrower's market price of Brownian motion risk from her wealth-weighted sum given by $\frac{1}{1+\phi(t)}\theta_b(t) + \frac{\phi(t)}{1+\phi(t)}\theta_l(t) = \sigma_e$. This leads to a borrower's market price of Brownian motion risk given by $\theta_b(t) = \sigma_e(1 + \phi(t) - \phi(t)\epsilon)$. The general form of the agent-specific pricing kernel given in equation 4 remains valid, the process for relative wealth in equation 10 generalizes to

$$\begin{aligned} \frac{d\phi(t)}{\phi(t-)} &= \left[(\lambda_b(t) - \lambda_{Q,b}(t)) - (\lambda_l(t) - \lambda_{Q,l}(t)) + \theta_l^2(t) - \theta_l(t)\theta_b(t)\right] dt \qquad (27) \\ &\quad + [\theta_l(t) - \theta_b(t)]\, dB(t) + \left(\frac{\lambda_l(t)\left(\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b(t)\right)}{\lambda_b(t)\left(\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l(t)\right)} - 1\right) dN(t, \lambda_i(t)), \end{aligned}$$

with $\lambda_{Q,b}(t)$ and $\lambda_{Q,l}(t)$ having the same interpretation as before.

Condition 1: This condition ensures that each agent's present value of the endowment stream equals $V(t) = e(t)(T-t)$. For consistency, $\phi(t)$ needs to satisfy the martingale property under the borrower's measure $\mathcal{P}_b$, and hence

$$\lambda_{Q,l}(t) - \lambda_{Q,b}(t) - \lambda_l(t) + \theta_l^2(t) - \theta_l(t)\theta_b(t) + \frac{\lambda_l(t)\left(\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b\right)}{\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l} = 0, \tag{28}$$

with $\theta_l(t)$ and $\theta_b(t)$ stipulated as above.

Condition 2: The rare event in the lender's wealth process has two components in case of default. The first one is generated from the FDI and equals the jump of the output process, the second is the write-off on the debt contract. Therefore, condition 2 generalizes to

$$\left(\frac{\lambda_l(t)}{\lambda_{Q,l}(t) - \kappa_D(t)\lambda_l(t)} - 1\right) = \epsilon\kappa_e + (1-\epsilon)\kappa_D(t). \tag{29}$$

Condition 3: In Section 2, this condition was given through the comparison of the Poisson terms of the borrower's state price density with the marginal utility of the RA. This is equivalent to determining the Poisson terms of the borrower's wealth process given his allocation toward Poisson risk. Hence, the equivalent to Condition 2 but for the borrowing agent including FDI yields

$$\left(\frac{\lambda_b(t)}{\lambda_{Q,b}(t) - \kappa_D(t)\lambda_b(t)} - 1\right) = ((1 + \phi(t) - \phi(t)\epsilon))\kappa_e + (\phi(t)\epsilon - \phi(t))\kappa_D(t). \tag{30}$$

Equations 28, 29, and 30 uniquely determine the write-off parameter $\kappa_D(t)$, and the two risk-adjusted default frequencies.

Condition 4: The remaining condition determines the rate of interest on the debt contract. As in the original economy, this can be done by comparing the deterministic terms of the borrower's state price density process with the marginal utility of the RA. Though the functional form of the state price density process remains unchanged, the deterministic term of the marginal utility process of equation 14 generalizes to

$$\frac{d\left[\frac{1+\phi(t)}{e(t)}\right]^{c}}{\frac{1+\phi(t-)}{e(t-)}} = \left[\begin{array}{c} \frac{\phi(t-)}{1+\phi(t-)}[\lambda_b(t) - \lambda_{Q,b}(t) - \lambda_l(t) + \lambda_{Q,l}(t) \\ +\theta_l^2(t) - \theta_l(t)\theta_b(t) - (\theta_l(t) - \theta_b(t))\sigma_e] - \widehat{\mu}_e + \sigma_e^2 \end{array}\right] dt + [...]. \tag{31}$$

We can numerically compute the equilibrium properties, and learn about the impact of FDI on the level of the interest rate and the write-off parameter. Overall, the interest rate can increase and decrease with FDI depending on the degree of heterogeneity in beliefs about default. The interest rate with FDI is higher for homogeneity or small levels of heterogeneity. This effect is driven by a smaller precautionary savings term in the interest rate, as there is more optimal risk sharing with FDI compared to no FDI with respect to Brownian motion risk. For a larger degree of heterogeneity this is still true, but this effect is overcompensated by a larger risk adjustment with respect to Poisson risk, leading to a lower interest rate on the debt contract. The impact on the level of the write-off parameter is ambiguous. Under homogeneity, and for the cases in which $\lambda_b(t) > \lambda_l(t)$ the effect is very small. This is not surprising as the lender even with a small FDI anticipates a lower likelihood for rare events, and is willing to write off a fraction larger than the shock to fundamentals, see Section 3. The impact is more severe for the case $\lambda_b(t) < \lambda_l(t)$. The lender participates directly in the shock κ_e through the FDI, and is willing to accept an even lower write-off on the debt contract compared to zero FDI.

We conclude this extension by re-calibrating the model to the case of Ecuador. According to Worldbank statistics, the average FDI as a percentage of GDP between 1970 and 2000 was 2% for Ecuador. In the base year of our calibration, 1997, Ecuador debt held externally was roughly 65% of GDP, hence we choose $\epsilon = .02/.65 = .03$. As before, we first extract the pairs of (dp_b,dp_l) that would generate a credit spread of 7.62%. Then we study the levels of implied write-off values in case of default, and the levels of implied credit spreads in pre-default times. The adjustments in calibration results compared to zero percent FDI are displayed in Figure 6.

The impact on write-off values is very small, between -.02% and .04%. For the region in which the borrower's belief is higher than the lenders belief, $\mathrm{dp}_b > \mathrm{dp}_l$, the write-off decreases. It increases for the opposite case of $\mathrm{dp}_b < \mathrm{dp}_l$. Under homogeneity, the write-off value is not affected by a small degree of FDI. The effect is slightly larger on the volatility of credit spreads. For example, for a realistic level of leverage in 1997, $\phi(t) = 6$, and given a lender's belief of 26%, a 3% FDI decreases the volatility in credit spreads by .45% from 10% to 9.55%. The higher the degree of leverage, the larger the effect due to FDI. Most interesting is the negative direction of adjustment in volatility. The intuition for this result is the effect of more complete risk sharing with respect to regular economic risk, and less limitation in participation. The higher the degree of FDI, the more the lender participates in regular risk of the economy, and the less volatile is the endogenous state variable $\phi(t)$ determined by $[\theta_l(t) - \theta_b(t)]$. Overall, our original results are fairly robust to the inclusion of a small FDI.

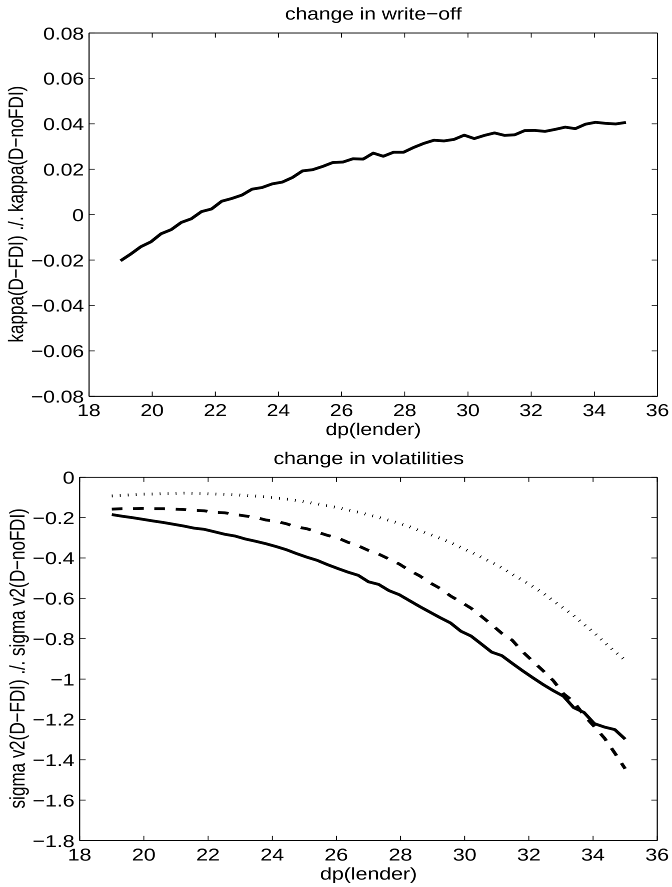

Figure 6: **Calibration results with FDI -** The upper graph shows the adjustment in implied write-off values due to the inclusion of 3% FDI, $\epsilon = .03$. The lower graph shows the adjustment in volatilities of credit spreads due to the inclusion of 3% FDI. Results are based on pairs of $(\mathrm{dp}_b, \mathrm{dp}_l)$ that generate a credit spread of 7.62% given $\kappa_e = -.24$ and $\sigma_e = .0625$. The solid, dashed, and dotted lines correspond to $\phi(t) = 6$, 4, and 2, respectively.

7.2 Proofs

Proof of Proposition 2. The deflated process for $D(t)$ is given by

$$d(\eta_i(t)D(t)) = D(t)d\eta_i(t)^c + \eta_i(t)dD(t)^c + (\eta_i(t)D(t) - \eta_i(t-)D(t-))dN(t, \lambda_i(t)),$$

which simplifies to

$$\begin{aligned}\frac{d(\eta_i(t)D(t))}{\eta_i(t-)D(t-)} &= -\alpha(t)dt + (\lambda_i(t) - \lambda_{Q,i}(t))dt - \theta_i(t)dB(t)) \\ &\quad + \left(\frac{(\lambda_{Q,i}(t) - \kappa_D(t)\lambda_i(t))(\kappa_D(t)+1)}{\lambda_i(t)} - 1\right) dN(t, \lambda_i(t)).\end{aligned}$$

For the martingale property to be satisfied, we show that every increment step between s and t is zero in expectation, while imposing the value of $\alpha(t)$ specified in the definition of the pricing kernel:

$$\begin{aligned}E_{i,t}\left[\frac{d(\eta(t)D(t))}{\eta_i(t-)D(t-)}\right] &= -\alpha(t)dt - \lambda_{Q,i}(t)dt + (-\kappa_D(t)\lambda_i(t) + \lambda_{Q,i}(t))(\kappa_D(t)+1)dt \\ &= (-\alpha(t) - \kappa_D^2(t)\lambda_i(t) - \kappa_D(t)\lambda_i(t) + \lambda_{Q,i}(t)\kappa_D(t))dt = 0.\end{aligned}$$

■

Proof of Proposition 3. The result is immediate from Proposition 1 for the case of homogeneous beliefs, $\lambda_b(t) = \lambda_l(t)$. We first show the upper boundary $\kappa_D(t) < 0$. Suppose $\lambda_b(t) < \lambda_l(t)$, then the difference $\lambda_b(t) - \lambda_l(t) = \overline{\lambda}(t) < 0$, and since the jump size $\kappa_e < 0$, the square root term in equation (22) $\sqrt{4\phi(t)(1+\phi(t))\kappa_e\lambda_b(t)\overline{\lambda}(t) + \beta(t)^2} > |\beta(t)|$. As $\beta(t)$ is negative and $\phi(t)$ positive, the result $\kappa_D(t) < 0$ follows. For the case $\lambda_b(t) > \lambda_l(t)$, the difference $\overline{\lambda}(t) > 0$, resulting in $\sqrt{4\phi(t)(1+\phi(t))\kappa_e\lambda_b(t)\overline{\lambda}(t) + \beta(t)^2} < |\beta(t)|$, and again the result $\kappa_D(t) < 0$ follows. Proving the lower boundary is equivalent to showing $\sqrt{4\phi(t)(1+\phi(t))\kappa_e\lambda_b(t)\overline{\lambda}(t) + \beta(t)^2} < -2\phi(t)\overline{\lambda}(t) - \beta(t)$, assuming $\lambda_b(t) < \lambda_l(t)$. The rhs and lhs are equal for homogeneous beliefs, and since the gradient for the lhs is negative at $\lambda_b(t)$, and the gradient for the rhs positive at $\lambda_b(t)$, the result follows since $\lambda_b(t) > \lambda_l(t)$. Assuming the reverse, $\lambda_b(t) > \lambda_l(t)$, the inequality reverses, as well as the direction of the gradients, and the desired result follows. ■

Proof of Proposition 4. The expression for $\kappa_{W_i}(t)$ is taken from equation (7), and the agent-specific values follow directly from equilibrium conditions, in particular that both agents together must share the entire exogenous jump size, $\frac{1}{1+\phi(t)}\kappa_{W_b}(t) + \frac{\phi(t)}{1+\phi(t)}\kappa_{W_l}(t) = \kappa_e$. The case of $\lambda_b(t) = \lambda_l(t)$, is trivial, as $\kappa_D(t) = \kappa_e$. If $\lambda_b(t) > \lambda_l(t)$, then $\kappa_D(t) < \kappa_e$, but since $\kappa_{W_l}(t) = \kappa_D(t)$ by construction, the result $\kappa_{W_b}(t) > \kappa_{W_l}(t)$ follows. The argument reverses for the case $\lambda_b(t) < \lambda_l(t)$. For a complete argument it remains to show that $\kappa_D(t) < \kappa_e$ in the case of $\lambda_b(t) > \lambda_l(t)$, and $\kappa_D(t) > \kappa_e$ in the case of $\lambda_b(t) < \lambda_l(t)$. Without loss of generality we can fix $\lambda_l(t)$, and determine the partial derivative of $\kappa_D(t)$ with respect to $\lambda_b(t)$, $\partial\kappa_D(t)/\partial\lambda_b(t)$. After simplifying the expression, the sign of $\partial\kappa_D(t)/\partial\lambda_b(t)$ is entirely determined by the term $-(\kappa_e(-1+\phi(t))+\phi(t))\lambda_b(t)+(-1+\kappa_e(-1+\phi(t)))\lambda_l(t)+\sqrt{4\phi(t)(1+\phi(t))\kappa_e\lambda_b(t)\overline{\lambda}(t)+(\beta(t))^2}$, which is strictly negative for all $\lambda_b(t) \neq \lambda_l(t)$, and $\phi(t) > 0$. Hence, the write-off in case of default $\kappa_D(t)$ is monotonically decreasing in $\lambda_b(t)$, and must be less negative than κ_e if $\lambda_b(t) < \lambda_l(t)$, and vice versa. The statement about relative wealth follows from $\frac{\kappa_{W_l}(t)+1}{\kappa_{W_b}(t)+1} - 1$, which is the jump size of the weighting process $\phi(t)$ in equation (10). ■