1. Let \(n\) be a positive integer. Prove using induction that
\[
\lim_{x \to \infty} \frac{x^n}{e^x} = 0.
\]

*Answer:* We first check that the statement is true for \(n = 1\), using l’Hôpital’s rule:
\[
\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.
\]

Next, we assume that
\[
\lim_{x \to \infty} \frac{x^k}{e^x} = 0
\]
and compute
\[
\lim_{x \to \infty} \frac{x^{k+1}}{e^x} = \lim_{x \to \infty} \frac{(k+1)x^k}{e^x} = (k+1) \lim_{x \to \infty} \frac{x^k}{e^x} = (k+1)(0) = 0.
\]
That completes the induction.

2. The Gamma function is defined by the formula
\[
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt
\]
for \(x \geq 1\). This is an improper integral, and for our purposes, you may assume that the integral converges with \(x \geq 1\). Prove that \(\Gamma(1) = 1\).

*Answer:* We have
\[
\Gamma(1) = \int_0^\infty e^{-t}dt = \lim_{t \to \infty} -e^{-t}\bigg|_0^t = 1 - \lim_{t \to \infty} \frac{1}{e^t} = 1 - 0 = 0.
\]

3. Use integration by parts, along with problem 1, to prove that \(\Gamma(n+1) = n\Gamma(n)\) if \(n\) is a positive integer.

*Answer:* For integration by parts, we set \(u = t^n\), \(dv = e^{-t}dt\), \(du = nt^{n-1}\), and \(v = -e^{-t}\). We have
\[
\Gamma(n+1) = \int_0^\infty t^n e^{-t}dt = -\left[t^n e^{-t}\right]_0^\infty + \int_0^\infty nt^{n-1}e^{-t}dt = 0 + n\Gamma(n-1).
\]
Here we compute
\[
-\left[t^n e^{-t}\right]_0^\infty = -\lim_{t \to \infty} \frac{t^n}{e^t} + \frac{0^n}{e^0} = 0 + 0 = 0
\]
by using problem 1.

4. Prove using induction that if \(n\) is a positive integer, then \(\Gamma(n) = (n-1)!\).

*Answer:* We know that \(\Gamma(1) = 1\) and \(0! = 1\), so the equation is true when \(n = 1\).

Now, assuming that \(\Gamma(k) = (k-1)!\), we compute \(\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!\), which completes the induction.
5. Let $n$ be a positive integer. Show that

$$
\sum_{k=1}^{n} F_k = F_{n+2} - 1.
$$

*Answer:* We proceed by induction. When $n = 1$, the left-hand side of the equation is $F_1$, which is 1, and the right-hand side is $F_3 - 1 = 2 - 1 = 1$. The equation is true when $n = 1$.

Now, assume that

$$
\sum_{k=1}^{r} F_k = F_{r+2} - 1.
$$

We compute

$$
\sum_{k=1}^{r+1} F_k = \sum_{k=1}^{r} F_k + F_{r+1} = (F_{r+2} - 1) + F_{r+1} = F_{r+3} - 1.
$$

That completes the induction.