1. Decide if \( f : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/14\mathbb{Z} \) given by the formula \( f([x]_7) = [x^2]_{14} \) is a well-defined function. Be sure to explain your answer fully.

\textit{Answer:} This function is \textit{not} well-defined. We have \([1]_7 = [8]_7\). The formula gives \( f([1]_7) = [1]_{14} \), and \( f([8]_7) = [64]_{14} \), and \([1]_{14} \neq [64]_{14}\).

2. Let \( n \) be a positive integer. Show that \( g : \mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^{n+1}\mathbb{Z} \) defined by \( g([x]_{2^n}) = [x^2]_{2^{n+1}} \) is well-defined.

\textit{Answer:} Suppose that \([x]_{2^n} = [y]_{2^n}\). We need to prove that \([x^2]_{2^{n+1}} = [y^2]_{2^{n+1}}\). In the language of congruences, we are given \( x \equiv y \pmod{2^n} \), and we need to prove that \( x^2 \equiv y^2 \pmod{2^{n+1}} \).

If \( x \equiv y \pmod{2^n} \), then \( y = x + k2^n \), and so \( y^2 = (x + k2^n)^2 = x^2 + k2^{n+1} + k^22^{2n} = x^2 + 2^{n+1}(k + k2^{2n-1}) \equiv x^2 \pmod{2^{n+1}} \). In other words, \([y^2]_{2^{n+1}} = [x^2]_{2^{n+1}}\), which is the desired result.

3. Suppose that \( A \) is a finite set, \( f : A \to A \), and \( g : A \to A \). Suppose in addition that \( f \circ g : A \to A \) is a bijection. Prove that \( f \) and \( g \) are both bijections.

\textit{Answer:} Suppose that \( g(a_1) = g(a_2) \). Then \( f(g(a_1)) = f(g(a_2)) \). Because \( f \circ g \) is a bijection, we can conclude that \( a_1 = a_2 \). This shows that \( g \) must be an injection. But if \( g : A \to A \) and \( A \) is a finite set, then \( g \) must be a bijection.

We could use a similar argument to show that \( f \) must be a surjection, and therefore also a bijection. Alternatively, we can reason as follows: \( g \) is a bijection, so \( g^{-1} : A \to A \) exists and is also a bijection. We are given that \( f \circ g \) is a bijection, and therefore \( f \circ g \circ g^{-1} \) is a bijection. Because \( f \circ g \circ g^{-1} = f \), we conclude that \( f \) is a bijection.

4. Give an explicit example to show that the conclusion to the previous problem is \textit{false} if \( A \) is an infinite set. You need to tell me what you are using for the set \( A \), what the functions \( f \) and \( g \) are, and why neither \( f \) nor \( g \) are bijections.

\textit{Answer:} We want \( f \circ g \) to be a bijection, which requires \( g \) to be an injection, and \( f \) to be a surjection. We are asked for an example in which neither \( f \) nor \( g \) are bijections.

One possibility is to take \( A = \mathbb{Z} \) and \( g : \mathbb{Z} \to \mathbb{Z} \) and \( f : \mathbb{Z} \to \mathbb{Z} \) to be defined by the formulas:

\[
g(n) = \begin{cases} 
  n + 1 & n \geq 0 \\
  n & n < 0 
\end{cases} 
\quad f(n) = \begin{cases} 
  n - 1 & n \geq 0 \\
  n & n < 0 
\end{cases}
\]

Then \( f \circ g(n) = n \). The function \( g \) is not a surjection, because there is no solution to \( g(n) = 0 \), and the function \( f \) is not an injection, because \( f(0) = f(-1) \). Notice, incidentally, that \( g \circ f(0) \neq 0 \), even though \( f \circ g(0) = 0 \).