1. Suppose that $m$, $n$, and $k$ are positive integers, with $(m, n) = 1$, $m|k$, and $n|k$. Prove that $mn|k$.

2. Suppose that $G$ is an abelian group, with $a, b \in G$. Suppose that $o(a) = m$ and $o(b) = n$, and $(m, n) = 1$. Prove that $o(ab) = mn$. Note: It is clear that $(ab)^{mn} = e$; the point is that you must show that no smaller exponent $j$ satisfies $(ab)^j = e$.

3. Suppose that $G$ is a finite abelian group, and $o(G) = p^a m$, where $p \nmid m$, $a \geq 1$, and $p$ is a prime. Let $H = \{ g \in G : g^{p^a} = e \}$.
   (a) Prove that $H$ is a subgroup of $G$.
   (b) Prove that if $h \in H$, then the only prime that might divide $o(h)$ is $p$.
   (c) Prove that the only prime dividing $o(H)$ is $p$. Hint: Apply Cauchy’s Theorem.
   (d) Show that $p \nmid o(G/H)$. Hint: Cauchy’s Theorem says that if $p|o(G/H)$, then $G/H$ contains a coset of order $p$. Now use an argument similar to the one which we used to prove Cauchy’s Theorem.
   (e) Show that $o(H) = p^a$.

   This is a specific case of one of the Sylow Theorems, which apply to both abelian and non-abelian groups. The proof is much trickier in the case of non-abelian groups.

4. If $\phi : G_1 \to G_2$ is a surjective homomorphism, and $N \triangleleft G_1$, show that $\phi(N) \triangleleft G_2$. You may assume that $\phi(N)$ is a subgroup of $G_2$.

5. If $H$ is any subgroup of $G$, let $N(H)$ be defined by:
   $$N(H) = \{ a \in G \mid aH = Ha \}.$$ 

   Prove that:
   (a) $N(H)$ is a subgroup of $G$, and $N(H) \supset H$.
   (b) $H \triangleleft N(H)$.
   (c) If $K$ is a subgroup of $G$ such that $H \triangleleft K$, then $K \subset N(H)$.

   These facts combine to tell us that $N(H)$ is the largest subgroup of $G$ in which $H$ is normal. The group $N(H)$ is called the normalizer of $H$. 