1. Suppose that $G$ is a finite group with subgroups $A$ and $B$. Prove that $o(AB) = o(A)o(B)/o(A \cap B)$. Note that typically, $AB$ will just be a subset of $G$ and not a subgroup.

2. If $(m,n) = 1$, show that the only group homomorphism $\phi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the trivial homomorphism. Remember that the group operation is addition.

3. Find a non-trivial group homomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.

4. Find a non-trivial group homomorphism $\phi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

5. Remember that the Hamiltonians $H$ are defined by $H = \{x_1 + ix_2 + jx_3 + kx_4 : x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ with $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$. Show there are infinitely many elements $x \in H$ satisfying $x^2 = -1$.

6. If $R$, $S$ are rings, define the direct sum of $R$ and $S$, $R \oplus S$, by

$$ R \oplus S = \{(r, s) : r \in R, s \in S\} $$

where $(r, s) = (r_1, s_1)$ if and only if $r = r_1$ and $s = s_1$, and where we define

$$(r, s) + (t, u) = (r + t, s + u), \quad (r, s)(t, u) = (rt, su).$$

(a) Show that $R \oplus S$ is a ring.
(b) Show that $\{(r, 0) : r \in R\}$ and $\{(0, s) : s \in S\}$ are ideals of $R \oplus S$.
(c) Show that $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is ring isomorphic to $\mathbb{Z}/6\mathbb{Z}$.
(d) Show that $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not ring isomorphic to $\mathbb{Z}/4\mathbb{Z}$. 