1. Suppose that $F$ is a field. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$$

$$I = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \mid d \in F \right\}$$

Show that

(a) $R$ is a ring.

(b) $I$ is an ideal of $R$.

(c) The function $\phi : R \to F \oplus F$ defined by $\phi \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = (a, c)$ is a ring homomorphism with kernel $I$.

2. Let $p$ be a prime. Show that the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

3. Suppose that $K$ and $L$ are two fields, with $K \subset L$. Suppose that $\dim_K(L) = n$. Let $a \in K$. Show that there are elements $\alpha_0, \alpha_1, \ldots, \alpha_n$ of $K$, not all zero, so that $\sum_{k=0}^{n} \alpha_k a^k = 0$.

4. Let $F$ be a field, let $f(x) \in F[x]$ be an irreducible polynomial, and suppose $\deg(f) = n \geq 1$. Let $M = (f(x))$, and let $K = F[x]/M$. We know that $K$ is a field containing $F$. Show that $\dim_F(K) = n$.

5. Suppose that $F$ is a field, $R$ is a ring, and $\phi : F \to R$ is a surjective ring homomorphism. Show that $\phi$ is a bijection, and that $R$ is a field.

6. Show that $\mathbb{R} \oplus \mathbb{R}$ is not ring isomorphic to $\mathbb{C}$.