1. Suppose that $X$ is a Poisson random variable with parameter $\lambda$. Show that $E[X^n] = \lambda E[(X+1)^{n-1}]$. Use this result to compute $E[X]$, $E[X^2]$, and $E[X^3]$.

**Answer:** We have

$$E[X^n] = \sum_{k=0}^{\infty} k^n \cdot P(X = k) = \sum_{k=0}^{\infty} k^n \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k^{n-1} \cdot e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=1}^{\infty} k^{n-1} \cdot e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=0}^{\infty} (k+1)^{n-1} \cdot e^{-\lambda} \frac{\lambda^k}{k!} = \lambda E[(X+1)^{n-1}]$$

Now we have $E[X] = \lambda E[1] = \lambda$, and then $E[X^2] = \lambda E[(X+1)^1] = \lambda E[X+1] = \lambda(\lambda+1) = \lambda^2 + \lambda$, and finally $E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2 + 2X + 1] = \lambda((\lambda^2 + \lambda) + 2\lambda + 1) = \lambda(\lambda^2 + 3\lambda + 1) = \lambda^3 + 3\lambda^2 + \lambda$.

2. (a) Write the power series expansion for $e^\lambda + e^{-\lambda}$.

(b) Suppose that $X$ is a Poisson random variable with parameter $\lambda$. Show that

$$P(X \text{ is even}) = (1 + e^{-2\lambda})/2.$$  

**Answer:** (a) We have

$$e^{\lambda} + e^{-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!} = 2 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}.$$

(b) We have

$$P(X \text{ is even}) = \sum_{k=0}^{\infty} P(X = 2k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} = e^{-\lambda} \left( \frac{e^{\lambda} + e^{-\lambda}}{2} \right) = \frac{1 + e^{-2\lambda}}{2}.$$

3. Whenever Alice and Bob have lunch, they decide who will pay for the meal by each rolling a die. The one who rolls the lower number must pay. If there is a tie, they roll again, and continue rolling until one rolls a lower number than the other.

What is the expected number of rolls needed to determine who will pay for lunch?

**Answer:** The probability of a tie is $\frac{1}{6}$, so the probability that the game ends in a single roll is $\frac{5}{6}$. Because this problem describes a geometric distribution, we know that the expected value is $\frac{1}{p} = \frac{6}{5}$.

4. Suppose that $X$ is a continuous random variable with probability density function

$$f(x) = \begin{cases} 
    c(1-x^4) & -1 < x < 1 \\
    0 & \text{otherwise}
\end{cases}$$
(a) What is $c$?
(b) What is the cumulative distribution function for $X$?

*Answer: (a) We compute*

$$
\int_{-1}^{1} (1 - x^4) \, dx = x - \frac{x^5}{5} \bigg|_{-1}^{1} = \frac{8}{5}.
$$

Therefore, $c = \frac{5}{8}$.

(b) If we denote the cumulative distribution function as $F(a)$, we know that $F(a) = 0$ for $a < -1$ and $F(a) = 1$ for $a > 1$. For $-1 \leq a \leq 1$, we have

$$
F(a) = \frac{5}{8} \int_{-1}^{a} (1 - x^4) \, dx = \frac{5}{8} \left( x - \frac{x^5}{5} \right)_{-1}^{a} = \frac{5}{8} \left( a - \frac{a^5}{5} + \frac{4}{5} \right).
$$

5. Suppose that $X$ is a binomial random variable with $E[X] = 6$ and $\text{Var}(X) = 2.4$. Compute $P(X = 5)$.

*Answer: We have $np = 6$ and $np(1-p) = 2.4$. Division yields $1-p = \frac{2.4}{6} = 0.4$, $p = 0.6$, and $n = \frac{6}{p} = 10$. Now, $P(X = 5) = \binom{10}{5} 0.6^5 0.4^5 \approx 0.2007$.

6. A filling station is provided with gasoline once each week. Suppose that the weekly volume of sales in thousands of gallons is a random variable with probability density function

$$
f(x) = \begin{cases} 
6(1-x)^5 & 0 < x < 1 \\
0 & \text{otherwise}
\end{cases}
$$

What must the capacity of the filling station’s tank be so that the probability of the station running out of gas in a given week is 0.01?

*Answer: One way to think about this problem is in terms of the cumulative distribution function $F(a)$: we wish to find $a$ so that $F(a) = 0.99$. We know that for $a \leq 0 \leq 1$, we have

$$
F(a) = \int_{0}^{a} 6(1-x)^5 \, dx = -(1-x)^6 \bigg|_{0}^{a} = 1 - (1-a)^6.
$$

We then must solve $(1-a)^6 = 0.01$, or $1-a \approx 0.4642$, or $a \approx 0.5358$. This is the capacity of the station’s tank, in thousands of gallons.

7. Remember that a poker hand contains 5 random cards. Let $X$ be the number of suits in a hand. Find $P(X = k)$ for $k = 1, 2, 3, \text{ and } 4$. *Hint: See Ch. 2, self-help prob. 11, for a discussion of $P(X = 4)$.*

*Answer: The simplest to compute is $P(X = 1)$: pick a suit, and then the cards in that suit.

$$
P(X = 1) = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{33}{16660} \approx 0.0020.
$$

For $X = 2$, we can either use Inclusion–Exclusion, or else rely on the fact that a two-suited hand must have either 4-1 distribution or else 3-2 distribution. For Inclusion–Exclusion,
we compute the probability that the hand contains no more than 2 suits, and then subtract the one-suited hands:

\[
P(X = 2) = \binom{4}{2} \cdot \left( \frac{26}{52} \right)^2 - 2 \binom{4}{2} \cdot \left( \frac{13}{52} \right)^2 = \frac{143}{980} \approx 0.1459.
\]

Alternatively, we can compute the probability of the two different distributions mentioned above and add:

\[
P(X = 2) = 4 \cdot 3 \left( \frac{13}{4} \binom{13}{1} \binom{13}{2} + \binom{13}{2} \binom{13}{1} \binom{13}{2} \right) = \frac{143}{980}
\]

Inclusion–Exclusion is trickier when \(X = 3\) (see if you can work it out on your own), but fortunately, there are again only 2 possible distributions: 2-2-1 and 3-1-1:

\[
P(X = 3) = \frac{6 \cdot 2 \left( \binom{13}{2} \binom{13}{1} \binom{13}{1} + \binom{13}{3} \binom{13}{1} \binom{13}{1} \right)} {\left( \frac{52}{5} \right)} = \frac{4901}{8330} \approx 0.5884.
\]

Finally, for \(X = 4\), the simplest approach is one that the text does not offer. The only possible distribution is 2-1-1-1:

\[
P(X = 4) = \frac{4 \binom{13}{2} \binom{13}{1} \binom{13}{1} \binom{13}{1}} {\left( \frac{52}{5} \right)} = \frac{2197}{8330} \approx 0.2637.
\]

The 4 probabilities sum to 1, as they must.

8. Let \(Y\) be a random variable. Show that

\[
E[Y] = \int_{0}^{\infty} P(Y > y) \, dy - \int_{0}^{\infty} P(Y < -y) \, dy
\]

by showing that

\[
\int_{0}^{\infty} P(Y > y) \, dy = \int_{0}^{\infty} x f_{Y}(x) \, dx
\]

\[
\int_{0}^{\infty} P(Y < -y) \, dy = -\int_{-\infty}^{0} x f_{Y}(x) \, dx.
\]

In these formulæ, \(f_{Y}(x)\) is the probability density function for \(Y\).

**Answer:** I prefer to work in the other direction, assuming that the probability density function for \(Y\) is \(f(y)\) and starting with the formula for \(E[Y]\):

\[
E[Y] = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} x f(x) \, dx + \int_{0}^{\infty} x f(x) \, dx = \int_{0}^{\infty} x f(x) \, dx - \int_{-\infty}^{0} (-x) f(x) \, dx
\]

\[
= \int_{0}^{\infty} \int_{0}^{x} dy f(x) \, dx - \int_{-\infty}^{0} \int_{x}^{\infty} f(x) \, dx
\]

In the first integral, we have \(0 \leq y \leq x \leq \infty\), while in the second integral we have \(-\infty < x < y < 0\). Changing the order of integration:

\[
= \int_{0}^{\infty} \int_{y}^{\infty} f(x) \, dx \, dy - \int_{-\infty}^{0} \int_{-\infty}^{y} f(x) \, dx \, dy = \int_{0}^{\infty} P(Y > y) \, dy - \int_{-\infty}^{0} P(Y < y) \, dy
\]
One more step: let $y = -z$ in the second integral, with $dy = -dz$:

$$
= \int_0^\infty P(Y > y) \, dy + \int_0^\infty P(Y < -z) \, dz = \int_0^\infty P(Y > y) \, dy - \int_0^\infty P(Y < -z) \, dz.
$$

9. If $X$ is a random variable with density function $f(x)$, show that

$$
E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) \, dx.
$$

*Hint:* The previous exercise shows that

$$
E[g(X)] = \int_0^\infty P(g(X) > y) \, dy - \int_0^\infty P(g(X) < -y) \, dy.
$$

Now proceed as we did in class.

*Answer:* Write $f(x)$ as the probability density function for $X$. We have

$$
E[g(X)] = \int_0^\infty P(g(X) > y) \, dy - \int_0^\infty P(g(X) < -y) \, dy
$$

$$
= \int_0^\infty \int_{x: g(x) > y} f(x) \, dx \, dy - \int_0^\infty \int_{x: g(x) < -y} f(x) \, dx \, dy.
$$

Again, we switch the order of integration in each of the integrals, noting in the second integral that if $g(x) < -y$, then $-g(x) > y$:

$$
= \int_{x: g(x) > 0} \int_0^{g(x)} dy \, f(x) \, dx - \int_{x: g(x) < 0} \int_0^{-g(x)} dy \, f(x) \, dx
$$

$$
= \int_{x: g(x) > 0} g(x)f(x) \, dx - \int_{x: g(x) < 0} (-g(x))f(x) \, dx
$$

$$
= \int_{x: g(x) > 0} g(x)f(x) \, dx + \int_{x: g(x) < 0} g(x)f(x) \, dx
$$

$$
= \int_{x: g(x) \neq 0} g(x)f(x) \, dx
$$

Finally, notice that $\int_{x: g(x) = 0} g(x)f(x) \, dx = 0$, because the integrand is 0:

$$
= \int_{x: g(x) \neq 0} g(x)f(x) \, dx + \int_{x: g(x) = 0} g(x)f(x) \, dx = \int_{-\infty}^{\infty} g(x)f(x) \, dx