1. **5.1.1.** Let \((M, d)\) be a metric space. Define the maps \(d_1, d_2 : M \times M \to \mathbb{R}\) by \(d_1 := d/(1 + d)\) and \(d_2 = \min(1, d)\); i.e., for any \(x, y \in M\), \(d_1(x, y) = d(x, y)/(1 + d(x, y))\) and \(d_2(x, y) = \min\{1, d(x, y)\}\). Show that \(d_1\) and \(d_2\) are both metrics on \(M\) and that we have \(d_1 \leq d_2 \leq 2d_1\).

**Answer:** First, we show that \(d_1\) is a metric. Clearly, \(d_1(x, y) \geq 0\). If \(d_1(x, y) = 0\), then \(d(x, y) = 0\), so \(x = y\), and \(d_1(x, x) = 0\). Clearly, \(d_1(x, y) = d_1(y, x)\). It remains to show that the triangle inequality holds for \(d_1\).

We start with a trivial observation. The function \(f(t) = t/(1 + t)\) is strictly increasing for \(t \geq 0\). (One way to see this is to notice that \(f'(t) > 0\).) Therefore, if \(d(a, b) \leq d(c, d)\), then \(d_1(a, b) \leq d_1(c, d)\).

We need to show that \(d_1(x, z) \leq d_1(x, y) + d_1(y, z)\). We know that \(d(x, z) \leq d(x, y) + d(y, z)\). We can divide by \(1 + d(x, z)\), and then we have

\[
\frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(x, z)}.
\]

If \(d(x, z) \geq d(x, y)\) and \(d(x, z) \geq d(y, z)\), then

\[
\frac{d(x, y)}{1 + d(x, z)} + \frac{d(y, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} = d_1(x, y) + d_1(y, z),
\]

and therefore, we are done.

Suppose that \(d(x, z) < d(x, y)\). In that case, \(d_1(x, z) < d_1(x, y)\), and therefore \(d_1(x, z) < d_1(x, y) + d_1(y, z)\).

Finally, if \(d(x, z) < d(y, z)\), then \(d_1(x, z) < d_1(y, z)\) and therefore \(d_1(x, z) < d_1(x, y) + d_1(y, z)\). This shows that \(d_1\) satisfies the triangle inequality and therefore is a metric.

Second, we show that \(d_2\) is a metric. We clearly have \(d_2(x, y) \geq 0\), and \(d_2(x, y) = 0\) if and only if \(x = y\). It is also clear that \(d_2(x, y) = d_2(y, x)\). Again, the triangle inequality looms as the obstacle.

If \(d(x, z) \leq d(x, y)\), then \(d_2(x, z) = \min\{1, d(x, z)\} \leq \min\{1, d(x, y)\} = d_2(x, y) \leq d_2(x, y) + d_2(y, z)\).

Similarly, if \(d(x, z) \leq d(y, z)\), then \(d_2(x, z) = \min\{1, d(x, z)\} \leq \min\{1, d(y, z)\} = d_2(y, z) \leq d_2(x, y) + d_2(y, z)\).

We therefore need only worry about the case that \(d(x, z) > d(x, y)\) and \(d(x, z) > d(y, z)\).

If \(d(x, z) \leq 1\), then \(d_2(x, z) = d(x, z)\), \(d_2(x, y) = d(x, y)\) and \(d_2(y, z) = d(y, z)\), so we are done. If \(d(x, z) > 1\), then \(d_2(x, z) = 1\). If \(d(x, y) > 1\) or \(d(y, z) > 1\), we are done, so we may assume that \(d(x, y) < 1\) and \(d(y, z) < 1\). In that case, we know that \(d(x, z) < d(x, y) + d(y, z)\), which implies that \(1 < d(x, y) + d(y, z) = d_2(x, y) + d_2(y, z)\). That exhausts all cases and shows that \(d_2\) is a metric.

We need to show that \(d_1 < d_2\). We know that \(d_1(x, y) < 1\) and \(d_1(x, y) < d_2(x, y)\), which shows that \(d_1(x, y) < \min\{1, d_2(x, y)\}\).

Finally, we need to show that \(d_2 \leq 2d_1\). Note that \(d_1(x, x) = 1 - \frac{1}{1 + d(x, x)} = \min\{1, d(x, x)\}\). If \(d(x, y) \geq 1\), then \(d_2(x, y) = 1\) while \(d_1(x, y) \geq \frac{1}{2}\), and therefore \(d_2 \leq 2d_1\).

If \(d(x, y) < 1\), then \(d_2(x, y) = d(x, y)\) while \(d_1(x, y) > d(x, y)/2\), allowing us to conclude that \(d_2 \leq 2d_1\).

2. **5.1.2.** Consider the set \(\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_k \in \mathbb{R} \text{ for } 1 \leq k \leq n\}\). Define the maps

\[
d_{\text{euc}}(x, y) := \sqrt[2]{\sum_{k=1}^{n} (x_k - y_k)^2},
\]

\[
d_{\text{max}}(x, y) := \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}
\]

\[
d_{\text{sum}}(x, y) := \sum_{k=1}^{n} |x_k - y_k|
\]

Show that \(d_{\text{euc}}, d_{\text{max}}, \text{ and } d_{\text{sum}}\) are metrics on \(\mathbb{R}^n\), and that we have the inequalities \(d_{\text{max}} \leq d_{\text{euc}} \leq d_{\text{sum}} \leq nd_{\text{max}}\).
Answer: To show that this theorem in fact is a general property of products of metric spaces, I will write $|x_k - y_k|$, as $d(x_k, y_k)$.

We easily see that $d_{\text{euc}}(x, y) \geq 0$, and that $d_{\text{euc}}(x, y) = 0$ if and only if $x = y$. The triangle inequality is not so clear.

We have

\[
d_{\text{euc}}(x, z) = \sqrt{\sum_{k=1}^{n} d(x_k, z_k)^2} \leq \sqrt{\sum_{k=1}^{n} (d(x_k, y_k) + d(y_k, z_k))^2} \\
\leq \sqrt{\sum_{k=1}^{n} d(x_k, y_k)^2} + \sqrt{\sum_{k=1}^{n} d(y_k, z_k)^2} = d_{\text{euc}}(x, y) + d_{\text{euc}}(y, z)
\]

The mysterious inequality at the start of the second line is the Triangle Inequality as stated on page 41.

We clearly have $d_{\text{max}}(x, y) \geq 0$ and $d_{\text{max}}(x, y) = 0$ if and only if $x = y$. Furthermore, it is immediate that $d_{\text{max}}(x, y) = d_{\text{max}}(y, z)$.

The triangle inequality is not too bad this time:

\[
d_{\text{max}}(x, z) = \max\{d(x_1, z_1), \cdots, d(x_n, z_n)\} \\
\leq \max\{d(x_1, y_1) + d(y_1, z_1), \cdots, d(x_n, y_n) + d(y_n, z_n)\} \\
\leq \max\{d(x_1, y_1), d(x_2, y_2)\} + \max\{d(y_1, z_1), d(y_2, z_2)\} = d_{\text{max}}(x, y) + d_{\text{max}}(y, z)
\]

Again, it’s easy to see that $d_{\text{sum}}(x, y) \geq 0$ and $d_{\text{sum}}(x, y) = 0$ if and only if $x = y$. It’s also trivial that $d_{\text{sum}}(x, y) = d_{\text{sum}}(y, x)$. And the triangle inequality this time is also easy:

\[
d_{\text{sum}}(x, z) = \sum_{k=1}^{n} d(x_k, z_k) \leq \sum_{k=1}^{n} (d(x_k, y_k) + d(y_k, z_k)) = \sum_{k=1}^{n} d(x_k, y_k) + \sum_{k=1}^{n} d(y_k, z_k) = d_{\text{sum}}(x, y) + d_{\text{sum}}(y, z).
\]

Finally, the inequalities:

\[
d_{\text{euc}}(x, y) = \sqrt{\sum_{k=1}^{n} d(x_k, y_k)^2} \\
\geq \max\{d(x_1, y_1), \cdots, d(x_n, y_n)\} = d_{\text{max}}(x, y) \\
d_{\text{sum}}(x, y)^2 = \left(\sum_{k=1}^{n} d(x_k, y_k)^2\right)^2 \\
\geq \sum_{k=1}^{n} d(x_k, y_k)^2 = d_{\text{euc}}(x, y)^2 \\
d_{\text{sum}}(x, y) = \sum_{k=1}^{n} d(x_k, y_k) \leq \sum_{k=1}^{n} \max\{d(x_k, y_k)\} = nd_{\text{max}}(x, y)
\]

3. 5.8.2. Let $M$ be a nonempty set and suppose that $d : M \times M \to \mathbb{R}$ satisfies the following conditions:

\[
d(x, y) = 0 \iff x = y \quad (\forall x, y \in M),
\]

\[
d(x, y) \leq d(x, z) + d(y, z) \quad (\forall x, y, z \in M)
\]

Show that $(M, d)$ is a metric space.

Answer: Apply the second equation with $y = x$, and we have $d(x, x) \leq d(x, z) + d(x, z)$. Because $d(x, x) = 0$, we see that $2d(x, z) \geq 0$, so $d(x, z) \geq 0$.

Apply the second equation with $z = x$, and we get $d(x, y) \leq d(x, x) + d(y, x)$. Because $d(x, x) = 0$, we have $d(x, y) \leq d(y, x)$. But $x$ and $y$ are arbitrary, so we can also deduce that $d(y, x) \leq d(x, y)$, implying that $d(x, y) = d(y, x)$.

4. 5.8.3. (a) Let $\ell^\infty(\mathbb{N})$ denote the set of all bounded real sequences $x \in \mathbb{R}^\mathbb{N}$. For each $x, y \in \ell^\infty(\mathbb{N})$, define

\[
d_{\infty}(x, y) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}.
\]

Show that $(\ell^\infty(\mathbb{N}), d_{\infty})$ is a metric space.
(b) Let $\ell^1(\mathbb{N})$ denote the set of all real sequences $x \in \mathbb{R}^\mathbb{N}$ that are *summable*, i.e., $\sum_{n=1}^{\infty} |x_n| < \infty$. For each $x, y \in \ell^1(\mathbb{N})$, define

$$d_1(x, y) := \sum_{n=1}^{\infty} |x_n - y_n|.$$  

Show that $(\ell_1(\mathbb{N}), d_1)$ is a metric space.

(c) Consider the space $\ell^2(\mathbb{N})$ of all real sequences $x \in \mathbb{R}^\mathbb{N}$ that are *square summable*, i.e., $\sum_{n=1}^{\infty} |x_n|^2 < \infty$.

For each $x, y \in \ell^2(\mathbb{N})$, define

$$d_2(x, y) := \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$  

Show that $(\ell^2(\mathbb{N}), d_2)$ is a metric space.

**Answer:** (a) First, note that because $\sup |x_n - y_n| \leq \sup |x_n| + \sup |y_n|$, we know that $d_\infty$ is defined. It is clear that $d_\infty(x, y) \geq 0$ and $d_\infty(x, y) = 0$ if and only if $x = y$. Because $\sup(A_n + B_n) \leq \sup A_n + \sup B_n$, we have

$$d_\infty(x, z) = \sup |x_n - z_n| = \sup |(x_n - y_n) + (y_n - z_n)|$$

$$\leq \sup |(x_n - y_n) + |y_n - z_n| \leq \sup |x_n - y_n| + \sup |y_n - z_n| = d_\infty(x, y) + d_\infty(y, z).$$

(b) Because $\sum |x_n - y_n| \leq \sum |x_n| + \sum |y_n|$, we know that $d_1(x, y)$ is defined. Clearly, $d_1(x, y) = d_1(y, x)$, $d_1(x, y) \geq 0$, and $d_1(x, y) = 0$ if and only if $x = y$. Finally,

$$d_1(x, z) = \sum_{n=1}^{\infty} |x_n - z_n| = \sum_{n=1}^{\infty} |(x_n - y_n) + (y_n - z_n)| \leq \sum_{n=1}^{\infty} |x_n - y_n| + |y_n - z_n|$$

$$= \sum_{n=1}^{\infty} |x_n - y_n| + \sum_{n=1}^{\infty} |y_n - z_n| = d_1(x, y) + d_1(y, z)$$

(c) The triangle inequality says that

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}$$

We therefore have

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}$$

and hence

$$\sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}$$

Therefore, $d_2(x, y)$ is defined. Again, we immediately see that $d_2(x, y) \geq 0$, that $d_2(x, y) = 0$ if and only if $x = y$, and that $d_2(x, y) = d_2(y, x)$. For the triangle inequality for $d_2$, we start with the usual triangle inequality:

$$\sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2}$$

First, we can let $n \to \infty$ on the right-hand side of the inequality to get

$$\sqrt{\sum_{i=1}^{\infty} (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{\infty} (y_i - z_i)^2}$$

and now letting $n \to \infty$ on the left-hand side shows that $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$. 