1. **8.7.3.** Show that \((\sin nx/(1 + nx))\) converges uniformly on \([a, \infty)\) for any \(a > 0\) but not for \(a = 0\).

**Answer:** Let \(f_n(x) = \sin nx/(1 + nx)\). Because \(|f_n(x)| < 1/(1 + nx)\), we see that if \(x \neq 0\), then \(\lim_{n \to \infty} f_n(x) = 0\).

Of course, \(f_n(0) = 0\), so we have \(\lim_{n \to \infty} f_n(x) = 0\) for all values of \(x\).

Now, if \(x \in [a, \infty)\), we have \(|f_n(x)| < 1/(1 + na)\). Given \(\epsilon > 0\), we can take \(N\) large enough so that \(1/(1 + Na) < \epsilon\), and then we have \(|f_n(x)| < \epsilon\) for all \(n > N\) for all \(x \in [a, \infty)\).

Suppose instead we consider \(x \in [0, \infty)\). For any positive integer \(n\), we can let \(x = 1/n\), and we have \(|f_n(1/n)| = (\sin 1)/2 > 0.4\). This shows that \(f_n(x)\) does not tend to 0 uniformly on \([0, \infty)\).

2. **8.7.4.** Let \(f_n(x) = (1 + x/n)^n\) for all \(x \in \mathbb{R}\) and \(n \in \mathbb{N}\). Show that \(f_n(x)\) converges uniformly to \(e^x\) on any compact interval \([a, b] \subset \mathbb{R}\).

**Answer:** First, we need to show that \(f_n(x) \to e^x\). Write \(f_n(x) = \exp(n \log(1 + x/n)) = \exp(n(\log(n) - x - \log n))\). Now, applying l'Hôpital's Rule, we have

\[
\lim_{n \to \infty} n(\log(n + x) - \log n) = \lim_{n \to \infty} \frac{\log(n + x) - \log n}{n^{-1}} = \lim_{n \to \infty} \frac{1/n + x}{-n^{-2}} = \lim_{n \to \infty} \frac{n^2 x}{n^2 + nx} = x,
\]

and so \(\lim \exp(n(\log(n + x) - \log n)) = e^x\).

It remains to show that the convergence is uniform on any compact interval \([a, b]\). To do that, we can use Dini’s Theorem (Theorem 8.2.2), and show that \(f_n(x) \leq f_{n+1}(x)\). We apply the binomial theorem to each function:

\[
f_n(x) = \left(1 + \frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \frac{n(n-1)}{2}\left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{6}\left(\frac{x}{n}\right)^3 + \cdots
\]

\[
f_{n+1}(x) = \left(1 + \frac{x}{n+1}\right)^{n+1} = 1 + (n+1)\left(\frac{x}{n+1}\right) + \frac{(n+1)n}{2}\left(\frac{x}{n+1}\right)^2 + \frac{(n+1)n(n-1)}{6}\left(\frac{x}{n+1}\right)^3 + \cdots
\]

We see that term by term, the monomials in the expansion of \(f_{n+1}(x)\) are at least as large as those in the expansion of \(f_n(x)\), showing that \(f_n(x) \leq f_{n+1}(x)\). Note that this argument only works when \(x > 0\); a more subtle argument is needed for \(x < 0\).

3. **8.7.12.** For each \(n \in \mathbb{N}\), consider the function

\[
f_n(x) := \frac{x^n}{1 + x^{2n}} \quad (\forall x \in \mathbb{R})
\]

(a) Show that the sequence \((f_n)\) converges uniformly on \([a, b]\) if and only if \(|x| \neq 1\) for all \(x \in [a, b]\) (i.e., \([a, b]\) does not contain the points 1 and \(-1\)).

(b) Find all \(x \in \mathbb{R}\) for which the series \(\sum f_n(x)\) is convergent. Also, find the intervals on which the convergence is uniform.

**Answer:** Note that \(f_n(x) = f_n(1/x)\), a fact which we will exploit momentarily.

If \(x = 1\), then \(f_n(x) = \frac{1}{2}\), so \(\lim_{n \to \infty} f_n(1) = \frac{1}{2}\). If \(x = -1\), then \(f_{2n}(-1) = \frac{1}{2}\), while \(f_{2n+1}(-1) = -\frac{1}{2}\), so \(\lim_{n \to \infty} f_n(-1)\) does not exist. If \(|x| < 1\), then \(\left|\frac{x^n}{1 + x^{2n}}\right| < |x^n|\), so \(\lim_{n \to \infty} f_n(x) = 0\). If \(|x| > 1\), then \(|1/x| < 1\), so \(\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(1/x) = 0\).
Suppose that \([a, b]\) does not contain \(\pm 1\). If \(|a| < 1\) and \(|b| < 1\), and \(x \in [a, b]\), then \(|f_n(x)| < |x^n| < \max(|a|^n, |b|^n)\). Given \(\varepsilon > 0\), choose \(N\) so that \(|a|^N < \varepsilon\) and \(|b|^N < \varepsilon\). If \(n > N\), then \(|f_n(x)| < \varepsilon\).

If \(|a| > 1\) and \(|b| > 1\), \(|f_n(x)| < \max(|a|^{-n}, |b|^{-n})\). Given \(\varepsilon > 0\), choose \(N\) so that \(|a|^{-N} < \varepsilon\) and \(|b|^{-N} < \varepsilon\). If \(n > N\), then \(|f_n(x)| < \varepsilon\).

4. 8.7.22. Prove each equation.

(a) \[\int_1^2 \left( \sum_{n=1}^{\infty} ne^{-nx} \right) dx = \frac{e}{e^2 - 1}\]

(b) \[\int_0^\pi \left( \sum_{n=1}^{\infty} \frac{n \sin nx}{e^n} \right) dx = \frac{2e}{e^2 - 1} \]

Answer: (a) First, note that if \(x \in [1, 2]\), then \(ne^{-nx} < 2e^{-n}\). Because \(\sum_{n=1}^{\infty} 2e^{-n}\) converges, the Weierstrass M-test shows that \(\sum_{n=1}^{\infty} ne^{-nx}\) converges uniformly on \([1, 2]\), and therefore we can interchange the order of integration and summation. We have

\[
\int_1^2 \left( \sum_{n=1}^{\infty} ne^{-nx} \right) dx = \sum_{n=1}^{\infty} \left( \int_1^2 ne^{-nx} dx \right) = \sum_{n=1}^{\infty} -e^{-nx} \bigg|_{x=1}^{x=2} = \sum_{n=1}^{\infty} e^{-n} - e^{-2n} = \frac{e^{-1}}{1 - e^{-1}} - \frac{e^{-2}}{1 - e^{-2}} = \frac{1}{e - 1} - \frac{1}{e^2 - 1} = \frac{e}{e^2 - 1}.
\]

(b) Because \(|\frac{n \sin nx}{e^n}| < ne^{-n}\), and \(\sum ne^{-n}\) converges, we can apply the Weierstrass M-test to conclude that \(\sum_{n=1}^{\infty} \frac{n \sin nx}{e^n}\) converges uniformly, and hence we can interchange the orders of integration and summation. We have

\[
\int_0^\pi \left( \sum_{n=1}^{\infty} \frac{n \sin nx}{e^n} \right) dx = \sum_{n=1}^{\infty} \left( \int_0^\pi \frac{n \sin nx}{e^n} dx \right) = \sum_{n=1}^{\infty} \left( \cos (n \pi) \right) = \sum_{n=1}^{\infty} \left( 1 - \cos (n \pi) \right) = \sum_{n=1}^{\infty} \frac{2}{e^{2n-1}} = \frac{2e}{1 - e^{-2}} = \frac{2e}{e^2 - 1}.
\]

5. 8.7.24. Let \(f_n(x) := x/(1+n^2x^2)\) for all \(x \in [-1, 1]\). Show that \((f_n)\) converges uniformly to a differentiable function \(f\), but that \(f'_n(x)\) does not converge to \(f'(x)\) for all \(x \in [-1, 1]\).

Answer: Note that \(\lim_{n \to \infty} f_n(0) = 0\). If \(x \neq 0\), then \(|f_n(x)| < \frac{|x|}{n^2|x|^2} = \frac{1}{n^2|\cdot|}\), so we still have \(\lim_{n \to \infty} f_n(x) = 0\).

We need to show that the convergence is uniform.

Given \(\varepsilon > 0\), note that if \(|x| < \varepsilon\), then \(|f_n(x)| < |x| < \varepsilon\) for all \(n\). For \(|x| > \varepsilon\), pick \(N\) so that \(\frac{1}{N^2 \varepsilon} < \varepsilon\), i.e., \(N^2 > \frac{1}{\varepsilon^2}\) or \(N > \frac{1}{\varepsilon}\).

We have \(|f_n(x)| < \frac{|x|}{n^2|x|^2} = \frac{1}{n^2|\cdot|} \leq \frac{1}{N^2 \varepsilon} < \frac{1}{N^2 \varepsilon} < \varepsilon\). This shows that \(f_n(x)\) converges to 0 uniformly on \([-1, 1]\).

We have \(f'_n(x) = \frac{1+n^2x^2-2n^2x^2}{(1+n^2x^2)^2} = \frac{-n^2x^2}{(1+n^2x^2)^2}\). We see that \(f'_n(0) = 1\) for all \(n\), while if \(x \neq 0\), \(f'_n(x) \to 0\).